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FIXED POINT RESULTS FOR WEAK *S*-CONTRACTIONS VIA *C*-CLASS FUNCTIONS

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ABSTRACT

In this paper, we prove some fixed point results for weak *S*-contraction mappings in partially ordered 2-metric spaces via *C*-class function. Our results are different and more general to the usual methods in the literature.

Keywords: *C*-class function, Weak *S*-contraction, Fixed point.

INTRODUCTION

Banach (1922) proved an important contraction principle. Many authors have been interested in this principle. Kannan (1968) established some contraction conditions in complete metric spaces. Choudhury (2009) and Shukla (see Shukla & Tiwari, 2011) proved some fixed point theorems for weakly *C*-contraction and weakly *S*-contraction mappings in complete metric spaces. Gähler (1963) introduced the notion of a 2-metric. Later, many authors obtained some fixed point theorems (Dung & Hang, 2013; Iseki, 1976). On the other hand, Birgani et al. (2018) established the definition of weak *S*-contraction and they proved some fixed point theorems for weak *S*-contraction. In this paper, we generalize weak *S*-contraction using *C*-class function. We prove some fixed point results in partially ordered 2-metric space. We obtain more general results and extend some known results in the existing literature.

Definition 1.1. (Birgani et al., 2018) Let $(Z, \preceq)$ is a partially ordered set. A mapping $P: Z \to Z$ is said to be monotone non-decreasing if $x, y \in Z$, $x \preceq y$, then $Px \preceq Py$.

Definition 1.2. (Gähler, 1963) Let Z be a non-empty set and $d_z : Z \times Z \times Z \to \mathbb{R}$ be a map such that:

(i) For every pair of distinct point $x, y \in Z$, there exists a point $z \in Z$, such that $d_z(x, y, z) \neq 0$.

(ii) If at least two of three points x, y, z are the same, then $d_z(x, y, z) = 0$.

(iii) For all $x,y,z \in Z$, $d_z(x, y, z) = d_z(x, z, y) = d_z(y, x, z)$ $= d_z(y, z, x) = d_z(z, x, y) = d_z(z, y, x)$.

(iv) For all $x,y,z,p \in Z$, $d_z(x, y, z) \le d_z(x, y, p) + d_z(y, z, p) + d_z(z, x, p)$.

Then d_z is called a 2-metric on Z and (Z, d_z) is called a 2-metric space.

Definition 1.3. (Iseki, 1975) Let (Z, d_z) be a 2-metric space and $\{x_n\}$ be a sequence in Z

(i) A sequence $\{x_n\}$ is said to be convergent to x in (Z, d_z), denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d_z(x_n, x, a) = 0$, for all $a \in Z$.

(ii) A sequence $\{x_n\}$ is a Cauchy sequence if $\lim_{n,m\to\infty} d_z(x_n, x_m, a) = 0$, for all $a \in Z$.

(iii) The 2-metric space (Z, d_z) is called complete if every Cauchy sequence is a convergent.

Definition 1.4. (Birgani et al., 2018) Let $(Z, \preceq, d_z)$ be a partially ordered 2-metric space and $P: Z \to Z$ be a map. Then, P is a called a weak *S*-contractions if there exists $\Psi : [0,\infty)^3 \to [0,\infty)$ which is continuous and $\Psi(\omega, s, p) = 0$, if and only if $s = \omega = p = 0$ such that

$$d_z(Px, Py, a) \le \frac{1}{3}\left(d_z(x, Py, a) + d_z(y, Px, a) + d_z(x, y, a)\right) - \Psi\left(d_z(x, Py, a), d_z(y, Px, a), d_z(x, y, a)\right) \quad (1)$$

for all $x, y, a \in Z$ and $x \preceq y$ or $y \preceq x$.

Definition 1.5. (Ansari, 2014) Let $F : [0,\infty)^2 \to \mathbb{R}$ be mapping, F is said to be *C*-class function if it satisfies following conditions:

1) F is continuous,
2) $F(s, p) \le s$,
3) $F(s, p) = s$ implies that either $s = 0$ or $p = 0$, for all $s, p \in [0,\infty)$ and $F(0,0) = 0$.

MAIN RESULTS

In this section, we present our main theorem. Let $\mathcal{C}$ denote all *C*-class functions. In this paper we gives Theorem 2.1 that shows the broadering of Theorem 4 in Birgani et al. (2018) using the *C*-class functions.

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Now we give by the function F generalization of the condition (5) from Birgani et al. (2018) as

$$d_z(Px,Py)\le F(\frac{1}{3}\left[d_z(x,Py)+d_z(Px,y)+d_z(x,y)\right],$$
$$\Psi\left(d_z(x,Py),d_z(Px,y),d_z(x,y)\right)).$$

Theorem 2.1. Let $(Z,\preceq,d_z)$ be a partially ordered 2-metric space, F is element of $\mathcal{C}$ and $P:Z\to Z$ is a weak S-contraction such that

(i) P is continuous and non-decreasing.

(ii) There exists $y_0\in Z$ such that $y_0\preceq Py_0$.

Then P has a fixed point.

Proof: The first part of proof of this theorem is very similar to the proof of Theorem 4 in Birgani et al. (2018). In our case we using the C-class function.

If $y_0=Py_0$, then the proof is complete. Let $y_0\preceq Py_0$. Since P is a non decreasing, we get $y_0\preceq Py_0\preceq P^2y_0\preceq...\preceq P^ny_0\preceq...$ Now we take $y_{n+1}=Py_n$. Then, for all $n\ge1$ and for all $a\in Z$, since y_{n-1} and y_n are comparable and from (1), we obtain

$$\begin{aligned}
&d_z(y_{n+1},y_n,a)=d_z(Py_n,Py_{n-1},a)\\
&\le F(\frac{1}{3}\left[d_z(y_n,Py_{n-1},a)+d_z(y_{n-1},Py_n,a)+d_z(y_n,y_{n-1},a)\right],\\
&\Psi\left[d_z(y_n,Py_{n-1},a),d_z(y_{n-1},Py_n,a),d_z(y_n,y_{n-1},a)\right])\\
&=F(\frac{1}{3}\left[d_z(y_n,y_n,a)+d_z(y_{n-1},y_{n+1},a)+d_z(y_n,y_{n-1},a)\right],\\
&\Psi\left[d_z(y_n,y_n,a),d_z(y_{n-1},y_{n+1},a),d_z(y_n,y_{n-1},a)\right])\\
&=F(\frac{1}{3}\left[d_z(y_{n-1},y_{n+1},a)+d_z(y_n,y_{n-1},a)\right],\\
&\Psi(d_z(y_{n-1},y_{n+1},a),d_z(y_n,y_{n-1},a)))\\
&\le\frac{1}{3}\left[d_z(y_{n-1},y_{n+1},a)+d_z(y_n,y_{n-1},a)\right] \qquad (2)
\end{aligned}$$

By putting $a=y_{n-1}$ in (2), we obtain $d_z(y_{n+1},y_n,y_{n-1})\le 0$, that is

$$d_z(y_{n+1},y_n,y_{n-1})=0. \qquad (3)$$

It follow from (2) and (3)

$$\begin{aligned}
&d_z(y_{n+1},y_n,a)\le\frac{1}{3}\left[d_z(y_{n-1},y_n,a)+d_z(y_n,y_{n+1},a)\right.\\
&\left.+d_z(y_{n-1},y_n,y_{n+1})+d_z(y_n,y_{n-1},a)\right]\\
&=\frac{2}{3}d_z(y_{n-1},y_n,a)+\frac{1}{3}d_z(y_n,y_{n+1},a). \qquad (4)
\end{aligned}$$

That is

$$d_z(y_{n+1},y_n,a)\le d_z(y_{n-1},y_n,a). \qquad (5)$$

Hence $\{d_z(y_n,y_{n+1},a)\}$ is a decreasing sequence. So it is convergent sequence. Let

$$\lim_{n\to\infty} d_z(y_{n+1},y_n,a)=p. \qquad (6)$$

Passing to limit $n\to\infty$ in (4) and from (6), we obtain

$$p\le\frac{1}{3}\left[\lim_{n\to\infty} d_z(y_{n-1},y_{n+1},a)+p\right]\le\frac{1}{3}(p+p+p)=p.$$

That is

$$\frac{2}{3}p\le\frac{1}{3}\lim_{n\to\infty} d_z(y_{n-1},y_{n+1},a)\le\frac{2}{3}p.$$

That is

$$2p\le\lim_{n\to\infty} d_z(y_{n-1},y_{n+1},a)\le 2p.$$

Therefore

$$\lim_{n\to\infty} d_z(y_{n-1},y_{n+1},a)=2p. \qquad (7)$$

Letting $n\to\infty$ in (2) and from (6) and (7), we obtain

$$p\le F(\frac{1}{3}(2p+p),\Psi(0,2p,p))\le\frac{1}{3}(2p+p)=p. \qquad (8)$$

Then from (6), we have

$$\lim_{n\to\infty} d_z\left(y_{n+1},y_n,a\right)=0. \qquad (9)$$

From (5), we get if $d_z(y_{n-1},y_{n+1},a)=0$, then $d_z(y_n,y_{n+1},a)=0$. Since $d_z(y_0,y_1,y_0)=0$, we get $d_z(y_n,y_{n+1},y_0)=0$. Since $d_z(y_{m-1},y_m,y_m)=0$. Thus, we get

$$d_z(y_n,y_{n+1},y_m)=0,\ \text{for all } n\ge m-1. \qquad (10)$$

Let $0\le n<m-1$, then, we get $m-1\ge n+1$. From (10) we get $d_z(y_{m-1},y_m,y_{n+1})=d_z(y_{m-1},y_m,y_n)=0$. It implies that

$$\begin{aligned}
&d_z(y_n,y_{n+1},y_m)\le d_z(y_n,y_{n+1},y_{m-1})+d_z(y_{n+1},y_m,y_{m-1})\\
&+d_z(y_n,y_m,y_{m-1})=d_z(y_n,y_{n+1},y_{m-1}). \qquad (11)
\end{aligned}$$

Since $d_z(y_n,y_{n+1},y_{n+1})=0$ from (11), we obtain

$$d_z(y_n,y_{n+1},y_m)=0,\quad \text{for}\quad 0\le n<m-1. \qquad (12)$$

From (10) and (12), we get $d_z(y_n,y_{n+1},y_m)=0$, for all $n,m\in\mathbb{N}$. For all $\eta,\mu,\kappa\in\mathbb{N}$ with $\eta>\mu$ we get $d_z(y_{\eta-1},y_\eta,y_\mu)=d_z(y_{\eta-1},y_\eta,y_k)=0$. Therefore

$$\begin{aligned}
&d_z(y_\eta,y_\mu,y_k)\le d_z(y_\mu,y_\eta,y_{\eta-1})+d_z(y_\eta,y_k,y_{\eta-1})\\
&+d_z(y_k,y_\mu,y_{\eta-1})\le d_z(y_\mu,y_{\eta-1},y_k) \qquad (13)\\
&\le...\le d_z(y_\eta,y_\mu,y_k)=0.
\end{aligned}$$

Hence, for all $\eta,\mu,\kappa\in\mathbb{N}$, we get

$$d_z(y_\eta,y_\mu,y_\kappa)=0. \tag{14}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{y_n\}$ is not a Cauchy sequence. Then there exists $\xi>0$ for which we can find subsequence $\{y_{n(k)}\}$ and $\{y_{m(k)}\}$ where $n(k)$ is the smallest integer such that $n(k)>m(k)>k$ and $y_{n(k)-1},y_{m(k)-1}$ are comparable and

$$d_z(y_{n(k)},y_{m(k)},a)\geq\xi,\ \text{for all}\ k\in\mathbb{N}. \tag{15}$$

Therefore,

$$d_z(y_{n(k)-1},y_{m(k)},a)\geq\xi.$$

Then from (13), (14), (15), we get

$$\begin{aligned}\xi&\leq d_z(y_{n(k)},y_{m(k)},a)=d_z(Py_{n(k)-1},Py_{m(k)-1},a)\\&\leq F(\tfrac{1}{3}\big[d_z(y_{n(k)-1},Py_{m(k)-1},a)+d_z(y_{m(k)-1},Py_{n(k)-1},a)\\&+d_z(y_{m(k)-1},y_{n(k)-1},a)\big],\\&\Psi\big[d_z(y_{n(k)-1},Py_{m(k)-1},a),d_z(y_{m(k)-1},Py_{n(k)-1},a),\\&d_z(y_{m(k)-1},y_{n(k)-1},a)\big])\\&=F(\tfrac{1}{3}\big[d_z(y_{n(k)-1},y_{m(k)},a)+d_z(y_{m(k)-1},y_{n(k)},a)\\&+d_z(y_{m(k)-1},y_{n(k)-1}a)\big],\\&\Psi(d_z(y_{n(k)-1},y_{m(k)},a),d_z(y_{m(k)-1},y_{n(k)},a),\\&d_z(y_{m(k)-1},y_{n(k)-1},a))).\end{aligned} \tag{16}$$

Now, by using (13), (14) and (15), we obtain

$$\begin{aligned}\xi&\leq d_z(y_{n(k)},y_{m(k)},a)\leq d_z(y_{n(k)},y_{n(k)-1},a)\\&+d_z(y_{n(k)-1},y_{m(k)},a)+d_z(y_{n(k)},y_{m(k)},y_{n(k)-1}).\end{aligned} \tag{17}$$

Letting $k\to\infty$ in (17), from (9), we get

$$\xi\leq\lim_{k\to\infty}d_z(y_{n(k)},y_{m(k)},a)\leq\xi$$

and

$$\begin{aligned}\xi&\leq\lim_{k\to\infty}d_z(y_{m(k)-1},y_{n(k)},a)+\lim_{k\to\infty}d_z(y_{m(k)-1},y_{m(k)},a)\\&+\lim_{k\to\infty}d_z(y_{m(k)-1},y_{n(k)},y_{m(k)})\leq\xi.\end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned}&\lim_{k\to\infty}d_z(y_{m(k)},y_{n(k)},a)=\xi,\\&\lim_{k\to\infty}d_z(y_{m(k)},y_{n(k)-1},a)=\xi,\\&\lim_{k\to\infty}d_z(y_{m(k)-1},y_{n(k)},a)=\xi.\end{aligned} \tag{19}$$

Taking the limit as $k\to\infty$ in (16), using (19), from the continuity of Ψ, we get

$$\xi\leq F(\tfrac{1}{3}\big[\xi+\xi+\xi\big],\Psi(\xi,\xi,\xi))=F(\xi,\Psi(\xi,\xi,\xi))\leq\xi.$$

That is,

$$F(\xi,\Psi(\xi,\xi,\xi))=\xi.$$

From Definition 1.5, we get $\xi=0$ or $\Psi(\xi,\xi,\xi)=0$ which is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since Z is complete, there exists $\omega\in Z$ such that $\lim_{n\to\infty}y_n=\omega$. Then, it follows from the continuity of P, we have

$$\omega=\lim_{n\to\infty}y_{n+1}=\lim_{n\to\infty}Py_n=P\omega.$$

Therefore, ω is a fixed point of P. This completes the proof. □

In Birgani et al. (2018) the Theorem 6 represents the basis for defining the following Theorem for the case when F is element of $\mathcal{C}$.

Theorem 2.2. Assume that hypotheses from the previous theorem hold and there exists $\omega\in P$ that is comparable to u and v, for each $u,v\in Z$. Then P has a unique fixed point.

Proof. Let u,v be two fixed points of P. We consider the following two cases.

Case 1: If v is comparable to ω, then $P^nv=v$ is comparable to $P^n\omega=\omega$ for all $n\in\mathbb{N}$. Therefore, for all $a\in Z$, and F is element of $\mathcal{C}$, we have

$$\begin{aligned}&d_z(v,\omega,a)=d_z(P^nv,P^n\omega,a)\\&\leq F(\tfrac{1}{3}\big[d_z(P^{n-1}v,P^{n-1}\omega,a)+d_z(P^{n-1}\omega,P^{n-1}v,a)+d_z(v,\omega,a)\big],\\&\Psi(d_z(P^{n-1}v,P^{n-1}\omega,a),d_z(P^{n-1}\omega,P^{n-1}v,a),d_z(v,\omega,a))\\&=F(\tfrac{1}{3}\big[d_z(v,\omega,a)+d_z(\omega,v,a)+d_z(v,\omega,a)\big],\\&\Psi(d_z(v,\omega,a),d_z(\omega,v,a),d_z(v,\omega,a)))\leq d_z(v,\omega,a).\end{aligned}$$

That is

$$F(d_z(v,\omega,a),\psi(d_z(v,\omega,a),d_z(\omega,v,a),d_z(v,\omega,a)))=d_z(v,\omega,a).$$

Hence, from Definition 1.5, we have $d_z(v,\omega,a)=0$ or $\Psi(d_z(v,\omega,a),d_z(\omega,v,a),d_z(v,\omega,a)))=0$. Therefore, we obtain $v=\omega$.

Case 2: If v is not comparable to ω, then there exists $u\in Z$ comparable to v and ω. It shows that $P^nu=u$ is comparable to $P^nv=v$ and $P^n\omega=\omega$ for all $n\in\mathbb{N}$.

Therefore, for all $a\in Z$, we get

$$d_z(\omega,P^nu,a)=d_z(P^n\omega,P^nu,a)$$
$$\leq F(\frac{1}{3}\left[d_z(P^{n-1}\omega,P^nu,a)+d_z(P^{n-1}u,P^n\omega,a)+d_z(P^n\omega,P^nu,a)\right],$$
$$\Psi(d_z(P^{n-1}\omega,P^nu,a),d_z(P^{n-1}u,P^n\omega,a),d_z(P^n\omega,P^nu,a)))$$
$$=F(\frac{1}{3}\left[d_z(\omega,P^nu,a)+d_z(P^{n-1}u,\omega,a)+d_z(\omega,P^nu,a)\right],$$
$$\Psi(d_z(\omega,P^nu,a),d_z(P^{n-1}u,\omega,a),d_z(\omega,P^nu,a)))$$
$$\leq\frac{2}{3}d_z(\omega,P^nu,a)+\frac{1}{3}d_z(\omega,P^{n-1}u,a). \tag{20}$$

That is

$$d_z(\omega,P^nu,a)\leq d_z(\omega,P^{n-1}u,a).$$

Then we obtain $\lim_{n\to\infty}d_z(\omega,P^nu,a)=m.$ Passing to limit $n\to\infty$ in (20), from the continuity of $\Psi,$ we get

$$m\leq F(\frac{1}{3}\left[m+m+m\right],\Psi(m,m,m))\leq m.$$

Then, we obtain $F(m,\Psi(m,m,m))=m.$ From the Definition 1.5, we get $m=0$ or $\Psi(m,m,m)=0.$ That is $m=0$. Hence $\lim_{n\to\infty}P^nu=\omega$ and $\lim_{n\to\infty}P^nu=v.$ Finally, we obtain $v=\omega.$ The theorem is thus proved. □

Also, next Theorem represent an expansion of the Theorem 7 in Birgani et al. (2018) for the case when *F* is element of $\mathcal{C}$.

Theorem 2.3. Let $(Z,\preceq,d_z)$ be a partially ordered 2-metric space, *F* is element of C and $P:Z\to Z$ be a weak *S*-contraction such that

- *(i)* If $u\preceq v$ for all $u,v\in Z,$ then $Pu\succeq Pv$.
- *(ii)* There exists $\omega\in Z$ that is comparable to u and v, for each $u,v\in Z.$
- *(iii)* There exists $u_0\in Z$ with $u_0\preceq Pu_0$ or $u_0\succeq Pu_0$.

Then, $\inf\left\{d_z(u,Pu,a):u\in Z\setminus\{a\}\right\}=0,$ for all $a\in Z.$ In particular $\inf\left\{d_z(u,Pu,a):u\in Z\right\}=0.$

Proof: Bearing in mind the proof of the Theorem 7 in Birgani et al. (2018), we proved Theorem 2.3 taking into account that it is *F* is element of $\mathcal{C}$.

We consider the following two cases.

Case 1: Let $u_0\preceq Pu_0$. By the hypothesis (*i*), consecutive terms of the sequence $\left\{P^nu_0\right\}$ are comparable. For all $a\in Z,$ it follows (1) that

$$d_z(P^{n+1}u_0,P^nu_0,a)$$
$$\leq F(\frac{1}{3}\left[d_z(P^nu_0,P^nu_0,a)+d_z(P^{n-1}u_0,P^{n+1}u_0,a)\right.$$
$$\left.+d_z(P^{n+1}u_0,P^nu_0,a)\right],$$
$$\Psi(d_z(P^nu_0,P^nu_0,a),d_z(P^{n-1}u_0,P^{n+1}u_0,a),$$
$$d_z(P^{n+1}u_0,P^nu_o,a)))$$
$$=F(\frac{1}{3}\left[d_z(P^{n-1}u_0,P^{n+1}u_0,a)+d_z(P^{n+1}u_0,P^nu_0,a)\right],$$
$$\Psi(d_z(P^{n-1}u_0,P^{n+1}u_0,a),d_z(P^{n+1}u_0,P^nu_0,a)))$$
$$\leq\frac{1}{3}\left[d_z(P^{n-1}u_0,P^{n+1}u_0,a)+d_z(P^{n+1}u_0,P^nu_0,a)\right]$$
$$\leq\frac{1}{3}\left[d_z(P^{n-1}u_0,P^nu_0,a)+d_z(P^nu_0,P^{n+1}u_0,a)\right.$$
$$\left.+d_z(P^{n-1}u_0,P^nu_0,P^{n+1}u_0)+d_z(P^{n+1}u_0,P^nu_0,a)\right]. \tag{21}$$

We have $d_z(u_\eta,u_\mu,u_\kappa)=0$ for all $\eta,\mu,\kappa\in\mathbb{N}.$ Then from (21), we get

$$d_z(P^{n+1}u_0,P^nu_0,a)\leq\frac{1}{3}\left[d_z(P^{n-1}u_0,P^nu_0,a)\right.$$
$$\left.+\frac{2}{3}d_z(P^nu_0,P^{n+1}u_0,a)\right].$$

That is

$$d_z(P^{n+1}u_0,P^nu_0,a)\leq d_z(P^{n-1}u_0,P^nu_0,a).$$

Then there exists $\lim_{n\to\infty}d_z(P^{n+1}u_0,P^nu_0,a)=p.$ Therefore, according to the previous theorems, we have $p=0.$ Then $\lim_{n\to\infty}d_z(P^{n+1}u_0,P^nu_0,a)=p.$ Hence, $\inf\left\{d_z(u,Pu,a):u\in Z\right\}=0.$

Case 2: Let $u_0\succeq Pu_0$. It is shown as in Case 1. This completes the proof. □

Example 2.1. Let $Z=\left[0,\infty\right)$ and we define a 2-metric d_z on *Z* by

$$d_z(x,y,a)=\min\left\{\left|x-y\right|,\left|y-a\right|,\left|x-a\right|\right\}.$$

Then it is clear that (Z,d_z) is a complete 2-metric space. We define a partial order on *Z* as follows:

$x\preceq y$ if and only if $x=y$ for all $x,y\in Z.$

Then $(Z,\preceq,d_z)$ is a complete partially ordered 2-metric space. Define $P:Z\to Z$ by $Px=\frac{1}{8},$ $F:\left[0,\infty\right)^2\to\mathbb{R}$ by $F(s,p)=ks$ for $k\in(0,1)$ and $\Psi:\left[0,\infty\right)^3\to\left[0,\infty\right)$ by $\Psi(u,v,\omega)=\frac{u+v+\omega}{8}.$ For all $x,y\in Z$ with $x\preceq y$, we have

$$d_z(Px,Py,a)=0\leq\frac{1}{3}k(d_z(x,Py,a)$$
$$+d_z(y,Px,a)+d_z(x,y,a))$$
$$=F(\frac{1}{3}\left[d_z(x,Py,a)+d_z(y,Px,a)+d_z(x,y,a)\right],$$
$$\Psi(d_z(x,Py,a),d_z(y,Px,a),d_z(x,y,a))).$$

Hence P is a weak S-contraction. Then, all the conditions of Theorem 2.1 are satisfied. Therefore, $1/8$ is a unique fixed point of P. Moreover, the condition of Theorem 2.2 does not hold. That is, for this example, it is not necessary condition to show uniqueness of the fixed point.

Example 2.2. Let $Z=[0,\infty)$ and we define a 2-metric d_z on Z by

$$d_z(x,y,a)=\begin{cases}1 & if \quad x\neq y\neq a\\ 0 & if \quad otherwise\end{cases}$$

Then it is clear that (Z,d_z) is a complete 2-metric space. We define a partial order on Z as follows:

$x\preceq y$ if and only if $x=y$ for all $x,y\in Z$.

Then $(Z,\preceq,d_z)$ is a complete partially ordered 2-metric space. Define $P:Z\to Z$ by $Px=x$, $F:[0,\infty)^2\to \mathrm{R}$ by $F(s,p)=s-p$ for $k\in(0,1)$ and $\Psi:[0,\infty)^3\to[0,\infty)$ by $\Psi(u,v,\omega)=\dfrac{\max\{u,v,\omega\}}{4}$.

Clearly, P is a weak S-contraction and all the conditions of Theorem 2.1 are satisfied. Therefore, for all $x\in Z$, x is a fixed point of P. That is P has more than one fixed point.

Conclusion: By using C-class function, we prove more general fixed point results for weak S-contraction in partially ordered 2-metric space. Also, we show that uniqueness of the fixed point of weak S-contraction with necessary conditions. Finally, we give some examples to support our main theorem. Our results are more general than some known results in the existing literature.

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