Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 96 – 109
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doi:10.3906/mat-2004-30

Notes on product semisymmetric connection in a locally decomposable Riemannian space

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Received: 07.04.2020 • **Accepted/Published Online:** 04.10.2020 • **Final Version:** 21.01.2021

Abstract: The purpose of this paper is to investigate the product semisymmetric connection in a locally decomposable Riemannian space. The curvature tensors of this connection were considered. Some properties of almost product structure, some properties of torsion tensor of product semisymmetric connection and some relations between curvature tensors and almost product structure are given. Also, the paper checks a special case of such connection when its generator is a gradient vector.

Key words: Decomposable Riemannian space, product semisymmetric connection, torsion tensor

1. Introduction

In the last century, the locally decomposable Riemannian space was investigated by several authors: Tachibana [20], Yano [21], Prvanović [12]–[15], etc. In recent years, Pušić has dealt with these spaces, observing various connections in them [16]–[19].

In [20], Tachibana obtained the product projective curvature tensor and the product conformal curvature tensor. Later, in [11], Petrović obtained the product concircular curvature tensor and Prvanović, in [15], obtained the product conharmonic curvature tensor. In the aricle [14] relations between these four curvature tensors were found.

In [12] and [13], Prvanović defined product semisymmetric metric F-connection using almost product structure F_j^i and covariant vector τ_i and observed some curvature tensors with respect to it. Two curvature tensors of product semisymmetric connection were examined in article [12] and it has been shown that one of them is actually the product projective curvature tensor. In [8], the product semisymmetric nonmetric connection was observed. Very interesting connections are also golden and metallic semisymmetric connection that have been studied in a locally decomposable golden and metallic Riemannian manifold [1, 2].

These works give us the motivation to study the curvature tensors with respect to product semisymmetric connection and to examine their properties. Since the product semisymmetric connection is nonsymmetric connection, we can observe four kinds of covariant derivatives with respect to it, by which twelve curvature tensors of nonsymmetric connection are obtained. We will begin by looking at curvature tensor in a generalized form for nonsymmetric connection, and then we will study five linearly independent tensors.

Since the definition of locally decomposable Riemannian space is similar to the definition of generalized Kählerian space, here we will show that between almost product structure tensor and curvature tensors, relations

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2010 *AMS Mathematics Subject Classification:* 53C05, 53B20, 53C15, 58J60

similar to those in the generalized Kählerian space apply [9, 10].

Finally, we specify the special case of generator τ_i when it is a gradient vector and show the properties obtained in this case for the generalized curvature tensor.

2. Product semisymmetric connection

Let us consider an N-dimensional manifold $\mathcal{M}_N$ which admits a tensor field $F^i_j \neq \delta^i_j$ of type (1,1) and a positive definite Riemannian metric $ds^2 = g_{ij}dx^i dx^j$. The manifold $\mathcal{M}_N$ is called a locally decomposable Riemannian space if the conditions

$$F^i_p F^p_j = \delta^i_j, \tag{2.1}$$

$$g_{pq}F^p_i F^q_j = g_{ij},\ \ g^{pq}F^i_p F^j_q = g^{ij}, \tag{2.2}$$

$$F^i_{j;k} = 0, \tag{2.3}$$

are satisfied, where δ^i_j is Kronecker symbol and the semicolon ; followed by an index denotes covariant derivative with respect to the Levi–Civita connection Γ^i_{jk}. We call the tensor F^i_j the almost product structure tensor.

Based on the above properties for F^i_j, it is easy to verify that $F_{ij} = F_{ji}$, where $F_{ij} = g_{ip}F^p_j$. Based on the fact that metric tensor g_{ij} is covariantly constant and based on Equation (2.3), by differentiating expression $g_{ip}F^p_j = F_{ij}$ we obtain

$$F_{ij;k} = \left(g_{ip}F^p_j\right)_{;k} = g_{ip;k}F^p_j + g_{ip}F^p_{j;k} = 0. \tag{2.4}$$

A locally decomposable space $\mathcal{M}_N$ can be covered by a separating coordinate system which is observed with respect to the two subspaces $\mathcal{M}_{N_1}$ and $\mathcal{M}_{N_2}$, where $N = N_1 + N_2$. This means that the manifold $\mathcal{M}_N$ is locally the product of two spaces $\mathcal{M}_{N_1} \times \mathcal{M}_{N_2}$ (for more details see [12, 16, 21]).

In the separating coordinate system we have

$$F^i_j = \begin{bmatrix} \delta^\pi_\sigma & 0 \\ 0 & -\delta^x_y \end{bmatrix},\ \ F_{ij} = \begin{bmatrix} g_{\pi\sigma} & 0 \\ 0 & -g_{xy} \end{bmatrix},$$

where $\pi, \sigma = 1, 2, \ldots, N_1$, $x, y = N_1 + 1, N_1 + 2, \ldots, N_1 + N_2 = N$. Therefore $\varphi = F^p_p = N_1 - N_2$. We will assume that $N_1, N_2 > 2$.

The connection

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + T^i_{jk} \tag{2.5}$$

where

$$T^i_{jk} = \delta^i_j \tau_k - \delta^i_k \tau_j + F^i_j F^p_k \tau_p - F^i_k F^p_j \tau_p \tag{2.6}$$

and τ_i is a covariant decomposable vector, is called the product semisymmetric connection of the locally decomposable Riemannian space (full name in [13]: product semisymmetric metric F-connection, because it satisfies the conditions $g_{ij;k} = 0$ and $F^i_{j;k} = 0$). Tensor T^i_{jk} is a torsion tensor of product semisymmetric connection. Covariant vector τ_i is called the generator of the space.

Theorem 2.1 ([21]) *A necessary and sufficient condition for a covariant vector* τ_i *in a locally decomposable Riemannian space to be decomposable is*

$$F_i^p \tau_{p;j} = F_j^p \tau_{i;p}. \tag{2.7}$$

From here we see that for decomposable vector also applies the equation

$$F_i^p F_j^q \tau_{p;q} = F_j^q F_q^p \tau_{i;p} = \tau_{i;j}. \tag{2.8}$$

Since the product semisymmetric connection is nonsymmetric connection, four kinds of covariant derivatives can be observed with respect to it [4, 5]

$$t^{i_1\dots i_A}_{j_1\dots j_B\underset{1}{|}k} = t^{i_1\dots i_A}_{j_1\dots j_B,k} + \sum_{p=1}^{A} t^{i_1\dots i_{\alpha-1} p i_{\alpha+1}\dots i_A}_{j_1\dots j_B} \overline{\Gamma}^{i_\alpha}_{pk} - \sum_{p=1}^{B} t^{i_1\dots i_A}_{j_1\dots j_{\alpha-1} p j_{\alpha+1}\dots j_B} \overline{\Gamma}^{p}_{j_\alpha k}, \tag{2.9}$$

$$t^{i_1\dots i_A}_{j_1\dots j_B\underset{2}{|}k} = t^{i_1\dots i_A}_{j_1\dots j_B,k} + \sum_{p=1}^{A} t^{i_1\dots i_{\alpha-1} p i_{\alpha+1}\dots i_A}_{j_1\dots j_B} \overline{\Gamma}^{i_\alpha}_{kp} - \sum_{p=1}^{B} t^{i_1\dots i_A}_{j_1\dots j_{\alpha-1} p j_{\alpha+1}\dots j_B} \overline{\Gamma}^{p}_{k j_\alpha}, \tag{2.10}$$

$$t^{i_1\dots i_A}_{j_1\dots j_B\underset{3}{|}k} = t^{i_1\dots i_A}_{j_1\dots j_B,k} + \sum_{p=1}^{A} t^{i_1\dots i_{\alpha-1} p i_{\alpha+1}\dots i_A}_{j_1\dots j_B} \overline{\Gamma}^{i_\alpha}_{pk} - \sum_{p=1}^{B} t^{i_1\dots i_A}_{j_1\dots j_{\alpha-1} p j_{\alpha+1}\dots j_B} \overline{\Gamma}^{p}_{k j_\alpha}, \tag{2.11}$$

$$t^{i_1\dots i_A}_{j_1\dots j_B\underset{4}{|}k} = t^{i_1\dots i_A}_{j_1\dots j_B,k} + \sum_{p=1}^{A} t^{i_1\dots i_{\alpha-1} p i_{\alpha+1}\dots i_A}_{j_1\dots j_B} \overline{\Gamma}^{i_\alpha}_{kp} - \sum_{p=1}^{B} t^{i_1\dots i_A}_{j_1\dots j_{\alpha-1} p j_{\alpha+1}\dots j_B} \overline{\Gamma}^{p}_{j_\alpha k}, \tag{2.12}$$

where $t^{i_1\dots i_A}_{j_1\dots j_B}$ is an arbitrary tensor. Based on these four kinds of covariant derivatives, using Ricci identities, twelve curvature tensor are obtained, five of which are linearly independent [6]

$$\underset{1}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm;n} - T^i_{jn;m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}, \tag{2.13}$$

$$\underset{2}{\overline{R}}{}^i_{jmn} = R^i_{jmn} - T^i_{jm;n} + T^i_{jn;m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}, \tag{2.14}$$

$$\underset{3}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm;n} + T^i_{jn;m} - T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm} - 2T^p_{mn}T^i_{pj}, \tag{2.15}$$

$$\underset{4}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + T^i_{jm;n} + T^i_{jn;m} - T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm} + 2T^p_{mn}T^i_{pj}, \tag{2.16}$$

$$\underset{5}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}, \tag{2.17}$$

where R^i_{jmn} denotes Riemannian curvature tensor with respect to the Levi–Civita connection Γ^i_{jk}.

Theorem 2.2 *For the almost product structure* F^i_j *in the locally decomposable Riemannian space the following relations*

$$F^i_{j\underset{1}{|}k} = 0,\ F^i_{j\underset{2}{|}k} = 0, \tag{2.18}$$

$$F^i_{j\underset{3}{|}k} = 2F^p_j T^i_{pk},\ F^i_{j\underset{4}{|}k} = -2F^p_j T^i_{pk}, \tag{2.19}$$

are true, where $\underset{\vartheta}{|}$ *denoting the operator of covariant derivative of* ϑ*-th kind,* $\vartheta = 1,2,3,4$*, with respect to the product semisymmetric connection* $\overline{\Gamma}^i_{jk}$*.*

Proof Using covariant derivative of the first kind with respect to nonsymmetric connection $\overline{\Gamma}^i_{jk}$ we have

$$F^i_{j\underset{1}{|}k} = F^i_{j,k} + F^p_j \overline{\Gamma}^i_{pk} - F^i_p \overline{\Gamma}^p_{jk} \overset{2.5}{=} F^i_{j;k} + F^p_j T^i_{pk} - F^i_p T^p_{jk}. \tag{2.20}$$

By direct application of the torsion tensor (2.6) we obtain that

$$F^p_j T^i_{pk} = F^i_p T^p_{jk}, \tag{2.21}$$

applies. Now, by substituting equations (2.3) and (2.21) into (2.20), we obtain $F^i_{j\underset{1}{|}k} = 0$.

Similarly, we can prove that $F^i_{j\underset{2}{|}k} = 0$.

By covariant derivative of the third and fourth kind and using the equations (2.3) and (2.21) we can prove that (2.19). □

Multiplying the equation (2.21) with F^q_i and using property (2.1), we have

$$T^q_{jk} = F^p_j F^q_i T^i_{pk}, \tag{2.22}$$

and after substituting the index $i \to p,\ p \to q,\ q \to i$, we obtain

$$T^i_{jk} = F^i_p F^q_j T^p_{qk}. \tag{2.23}$$

Also, substituting the index j and k in equation (2.21) we obtain that $F^p_k T^i_{jp} = F^i_p T^p_{jk}$ applies, so now

$$F^p_j T^i_{pk} = F^p_k T^i_{jp} = F^i_p T^p_{jk}. \tag{2.24}$$

Equation (2.24) means that the torsion tensor of product semisymmetric connection is pure tensor, so we conclude that also the product semisymmetric connection is pure [7].

Using almost product structure tensor F^i_j the Nijenhuis tensor can be defined as follows:

$$N^i_{jk} = \left(F^i_{j,p} - F^i_{p,j}\right) F^p_k - \left(F^i_{k,p} - F^i_{p,k}\right) F^p_j, \tag{2.25}$$

i.e.

$$N^i_{jk} = F^i_{[j,p]} F^p_k - F^i_{[k,p]} F^p_j, \tag{2.26}$$

where the square brackets $[ij]$ denote alternation without division with respect to the indices i and j.

Theorem 2.3 *In a locally decomposable Riemannian space the Nijenhuis tensor vanishes.*

Proof Starting from the fact that in locally decomposable Riemannian space $F^i_{j;k}=0$ applies, we obtain

$$F^i_{j;k} = F^i_{j,k} + F^p_j\Gamma^i_{pk} - F^i_p\Gamma^p_{jk} = 0. \tag{2.27}$$

Interchanging j and k in the equation (2.27) we get the expression for $F^i_{k;j}$ and by subtracting the equations thus obtained, we have the relation

$$F^i_{[j,k]} = -F^p_j\Gamma^i_{pk} + F^p_k\Gamma^i_{pj}. \tag{2.28}$$

Substituting this equation into (2.26), we get that $N^i_{jk}=0$. □

The following theorem shows that the torsion tensor (2.6) of product semisymmetric connection satisfies Jacoby's identity. For Jacoby's identity you can see the article [3].

Theorem 2.4 *For torsion tensor*

$$T^i_{jk} = \delta^i_j\tau_k - \delta^i_k\tau_j + F^i_jF^p_k\tau_p - F^i_kF^p_j\tau_p,$$

Jacoby's identity applies, i.e. the following equation is satisfied

$$T^p_{jm}T^i_{pn} + T^p_{mn}T^i_{pj} + T^p_{nj}T^i_{pm} = 0. \tag{2.29}$$

Proof The proof is trivial. By directly applying the torsion tensor (2.6) to Equation (2.29), we confirm this theorem. □

3. Generalization of the curvature tensor

Let us observe the curvature tensor of nonsymmetric connection in a generalized form

$$\overline{R}^i_{jmn} = R^i_{jmn} + uT^i_{jm;n} + u'T^i_{jn;m} + vT^p_{jm}T^i_{pn} + v'T^p_{jn}T^i_{pm} + wT^p_{mn}T^i_{pj} \tag{3.1}$$

where $u,u',v,v',w\in\mathbb{R}$. Substituting the expression for the torsion tensor T^i_{jk} from Equation (2.6) into (3.1), we have

$$\begin{aligned}
\overline{R}^i_{jmn} = R^i_{jmn} &+ u\left(\delta^i_j\tau_m - \delta^i_m\tau_j + F^i_jF^p_m\tau_p - F^i_mF^p_j\tau_p\right)_{;n} + u'\left(\delta^i_j\tau_n - \delta^i_n\tau_j + F^i_jF^p_n\tau_p - F^i_nF^p_j\tau_p\right)_{;m}\\
&+ v\left(\delta^p_j\tau_m - \delta^p_m\tau_j + F^p_jF^q_m\tau_q - F^p_mF^q_j\tau_q\right)\left(\delta^i_p\tau_n - \delta^i_n\tau_p + F^i_pF^q_n\tau_q - F^i_nF^q_p\tau_q\right)\\
&+ v'\left(\delta^p_j\tau_n - \delta^p_n\tau_j + F^p_jF^q_n\tau_q - F^p_nF^q_j\tau_q\right)\left(\delta^i_p\tau_m - \delta^i_m\tau_p + F^i_pF^q_m\tau_q - F^i_mF^q_p\tau_q\right)\\
&+ w\left(\delta^p_m\tau_n - \delta^p_n\tau_m + F^p_mF^q_n\tau_q - F^p_nF^q_m\tau_q\right)\left(\delta^i_p\tau_j - \delta^i_j\tau_p + F^i_pF^q_j\tau_q - F^i_jF^q_p\tau_q\right).
\end{aligned}$$

After the necessary calculation and using (2.1), whereby it is taken into consideration $\delta^i_{j;k}=0$ and $F^i_{j;k}=0$, we obtain

$$\begin{aligned}
\overline{R}^i_{jmn} = R^i_{jmn} &+ \delta^i_j\left(u\tau_{m;n} + u'\tau_{n;m} + (v+v')\left(\tau_m\tau_n + F^p_mF^q_n\tau_p\tau_q\right)\right)\\
&- \delta^i_m\left(u\tau_{j;n} + (v-w)\left(\tau_j\tau_n + F^p_jF^q_n\tau_p\tau_q\right)\right)\\
&- \delta^i_n\left(u'\tau_{j;m} + (v'+w)\left(\tau_j\tau_m + F^p_jF^q_m\tau_p\tau_q\right)\right)\\
&+ F^i_j\left(uF^p_m\tau_{p;n} + u'F^p_n\tau_{p;m} + (v+v')\left(F^p_m\tau_n + F^p_n\tau_m\right)\tau_p\right)\\
&- F^i_m\left(uF^p_j\tau_{p;n} + (v-w)\left(F^p_j\tau_n + F^p_n\tau_j\right)\tau_p\right)\\
&- F^i_n\left(u'F^p_j\tau_{p;m} + (v'+w)\left(F^p_j\tau_m + F^p_m\tau_j\right)\tau_p\right).
\end{aligned} \tag{3.2}$$

Furthermore, it follows that

$$\begin{aligned}\overline{R}^i_{jmn} = R^i_{jmn} + \delta^i_j\left(\beta_{mn} + \gamma_{nm}\right) - \delta^i_m\beta_{jn} - \delta^i_n\gamma_{jm}\\ + F^i_j\left(F^p_m\beta_{pn} + F^p_n\gamma_{pm}\right) - F^i_mF^p_j\beta_{pn} - F^i_nF^p_j\gamma_{pm},\end{aligned} \tag{3.3}$$

where

$$\beta_{ij} = u\tau_{i;j} + (v - w)\left(\tau_i\tau_j + F^p_iF^q_j\tau_p\tau_q\right), \tag{3.4}$$

$$\gamma_{ij} = u'\tau_{i;j} + (v' + w)\left(\tau_i\tau_j + F^p_iF^q_j\tau_p\tau_q\right). \tag{3.5}$$

If we take

$$\alpha_{ij} = \beta_{ij} + \gamma_{ji}, \quad \varepsilon_{ij} = \zeta_{ij} + \eta_{ji}, \quad \zeta_{ij} = F^p_i\beta_{pj}, \quad \eta_{ij} = F^p_i\gamma_{pj}. \tag{3.6}$$

then the generalized curvature tensor $\overline{R}^i_{jmn}$ for product semisymmetric connection in a locally decomposable Riemannian space can be written in the form

$$\overline{R}^i_{jmn} = R^i_{jmn} + \delta^i_j\alpha_{mn} - \delta^i_m\beta_{jn} - \delta^i_n\gamma_{jm} + F^i_j\varepsilon_{mn} - F^i_m\zeta_{jn} - F^i_n\eta_{jm}. \tag{3.7}$$

Below we will list some of the properties for the introduced tensors α_{ij}, β_{ij}, γ_{ij}, ε_{ij}, ζ_{ij} and η_{ij}, which we will need to study the properties of curvature tensor of product semisymmetric connection.

Theorem 3.1 *The expressions for tensors β_{ij} and γ_{ij} satisfy the following relations*

$$F^p_i\beta_{pj} = F^p_j\beta_{ip}, \;\; F^p_i\gamma_{pj} = F^p_j\gamma_{ip}, \tag{3.8}$$

and

$$\beta_{ij} = F^p_iF^q_j\beta_{pq}, \;\; \gamma_{ij} = F^p_iF^q_j\gamma_{pq}. \tag{3.9}$$

Proof All relations prove similarly. We will prove that $\beta_{ij} = F^p_iF^q_j\beta_{pq}$.

By using Equation (3.4), we have

$$F^p_iF^q_j\beta_{pq} = uF^p_iF^q_j\tau_{p;q} + (v - w)\left(F^p_iF^q_j\tau_p\tau_q + F^p_iF^q_jF^r_pF^s_q\tau_r\tau_s\right). \tag{3.10}$$

Furthermore, given that τ_i is a decomposable vector (see Equation (2.8)) and using Equation (2.1), we obtain equation

$$F^p_iF^q_j\beta_{pq} = \beta_{ij}.$$

□

Theorem 3.2 *The expressions for tensors α_{ij}, β_{ij}, γ_{ij}, ε_{ij}, ζ_{ij} and η_{ij} satisfy the following relations*

$$\alpha_{ij} - \beta_{ji} - \gamma_{ij} = (u - u')\tau_{[i;j]}, \tag{3.11}$$

$$\varepsilon_{ij} - \zeta_{ji} - \eta_{ij} = (u - u')F^p_i\tau_{[p;j]}. \tag{3.12}$$

where the square brackets $[ij]$ denote alternation without division with respect to the indices i and j.

Proof Based on Equations (3.4)–(3.6) we can easily conclude that the equation (3.11) is valid. Further, by using the expressions for ε_{ij}, ζ_{ij} and η_{ij} we have

$$\varepsilon_{ij} - \zeta_{ji} - \eta_{ij} = (u - u')(F_i^p \tau_{p;j} - F_j^p \tau_{p;i}). \tag{3.13}$$

If we take into consideration the decomposability property of vector τ_i, i.e. $F_i^p \tau_{p;j} = F_j^p \tau_{i;p}$, we get the equation

$$\varepsilon_{ij} - \zeta_{ji} - \eta_{ij} = (u - u')(F_i^p \tau_{p;j} - F_i^p \tau_{j;p}) = (u - u')F_i^p \tau_{[p;j]}. \tag{3.14}$$

□

Now, using Equations (3.11) and (3.12), we can generalize the cyclic symmetry property of the curvature tensors (3.7).

Theorem 3.3 *Let $(\mathcal{M}_N, g, F)$ be a locally decomposable Riemannian space endowed with the product semisymmetric connection (2.5). For curvature tensor $\overline{R}^i_{jmn}$ the next relation*

$$\underset{jmn}{Cycl}\, \overline{R}^i_{jmn} = (u - u')\, \underset{jmn}{Cycl} \left(\delta^i_j \tau_{[m;n]} + F^i_j F^p_m \tau_{[p;n]} \right) \tag{3.15}$$

is valid, where $\underset{jmn}{Cycl}$ is the cyclic sum regarding to j, m, n.

Proof By using the expression for curvature tensor $\overline{R}^i_{jmn}$, we obtain

$$\begin{aligned} \underset{jmn}{Cycl}\, \overline{R}^i_{jmn} &= \overline{R}^i_{jmn} + \overline{R}^i_{mnj} + \overline{R}^i_{njm} \\ &= \underset{jmn}{Cycl} \left(\delta^i_j (\alpha_{mn} - \beta_{nm} - \gamma_{mn}) + F^i_j (\varepsilon_{mn} - \zeta_{nm} - \eta_{mn}) \right), \end{aligned}$$

where the cyclic symmetry (the first Bianchi identity) of Riemannian curvature tensor R^i_{jmn} is taken into consideration. Finally, with the help of Equations (3.11) and (3.12) we prove the theorem. □

3.1. First kind of curvature tensor

As noted at the beginning of this paper, with respect to nonsymmetric connection, we can observe five linearly independent curvature tensors. First, let us consider the curvature tensor of the first kind of nonsymmetric connection. From Equation (2.13) for this tensor, we have $u = 1$, $u' = -1$, $v = 1$, $v' = -1$, $w = 0$ and by substituting in Equation (3.4) we get

$$\underset{1}{\beta}_{ij} = \tau_{i;j} + \tau_i \tau_j + F_i^p F_j^q \tau_p \tau_q = -\underset{1}{\gamma}_{ij}. \tag{3.16}$$

From there we obtain the curvature tensor of the first kind of product semisymmetric connection in the locally decomposable Riemannian space

$$\underset{1}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + \delta^i_j \underset{1}{\beta}_{[mn]} - \delta^i_m \underset{1}{\beta}_{jn} + \delta^i_n \underset{1}{\beta}_{jm} + F^i_j \underset{1}{\beta}_{p[n} F^p_{\ m]} - F^i_m F^p_j \underset{1}{\beta}_{pn} + F^i_n F^p_j \underset{1}{\beta}_{pm}. \tag{3.17}$$

If we introduce the notations

$$\underset{1}{\alpha}_{ij} = \underset{1}{\beta}_{[ij]}, \ \underset{1}{\varepsilon}_{ij} = \underset{1}{\beta}_{p[j} F^p_{i]}, \ \underset{1}{\zeta}_{ij} = F^p_i \underset{1}{\beta}_{pj}, \tag{3.18}$$

then the tensor $\underset{1}{\overline{R}}{}^i_{jmn}$ can be written in the form

$$\underset{1}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + \delta^i_j \underset{1}{\alpha}_{mn} - \delta^i_m \underset{1}{\beta}_{jn} + \delta^i_n \underset{1}{\beta}_{jm} + F^i_j \underset{1}{\varepsilon}_{mn} - F^i_m \underset{1}{\zeta}_{jn} + F^i_n \underset{1}{\zeta}_{jm}. \tag{3.19}$$

From Equation (3.18), we can easily conclude that $\underset{1}{\alpha}_{ij}$ and $\underset{1}{\varepsilon}_{ij}$ are antisymmetric, i.e.

$$\underset{1}{\alpha}_{ij} = -\underset{1}{\alpha}_{ji}, \;\; \underset{1}{\varepsilon}_{ij} = -\underset{1}{\varepsilon}_{ji}. \tag{3.20}$$

3.2. Second kind of curvature tensor

By observing the curvature tensor of the second kind (2.13) of nonsymmetric connection we can see that is $u = -1,\ u' = 1,\ v = 1,\ v' = -1,\ w = 0$ and by substituting these parameters in Equation (3.4), we have the expressions for tensor $\underset{2}{\beta}_{ij}$ and that way we obtain the curvature tensor of the second kind of product semisymmetric connection in a locally decomposable Riemannian space

$$\underset{2}{\overline{R}}{}^i_{jmn} = R^i_{jmn} - \delta^i_j \underset{2}{\beta}_{[mn]} + \delta^i_m \underset{2}{\beta}_{jn} - \delta^i_n \underset{2}{\beta}_{jm} - F^i_j \underset{2}{\beta}_{p[n} F^p_{\;m]} + F^i_m F^p_j \underset{2}{\beta}_{pn} - F^i_n F^p_j \underset{2}{\beta}_{pm}, \tag{3.21}$$

i.e.

$$\underset{2}{\overline{R}}{}^i_{jmn} = R^i_{jmn} - \delta^i_j \underset{2}{\alpha}_{mn} + \delta^i_m \underset{2}{\beta}_{jn} - \delta^i_n \underset{2}{\beta}_{jm} - F^i_j \underset{2}{\varepsilon}_{mn} + F^i_m \underset{2}{\zeta}_{jn} - F^i_n \underset{2}{\zeta}_{jm}, \tag{3.22}$$

where we introduced notations

$$\underset{2}{\alpha}_{ij} = \underset{2}{\beta}_{[ij]}, \quad \underset{2}{\beta}_{ij} = \tau_{i;j} - \tau_i \tau_j - F^p_i F^q_j \tau_p \tau_q, \tag{3.23}$$

$$\underset{2}{\varepsilon}_{ij} = \underset{2}{\zeta}_{[ij]} = \underset{2}{\beta}_{p[n} F^p_{\;m]}, \quad \underset{2}{\zeta}_{ij} = F^p_i \underset{2}{\beta}_{pj}. \tag{3.24}$$

It is clear from these equations that the tensors $\underset{2}{\alpha}_{ij}$ and $\underset{2}{\varepsilon}_{ij}$ are antisymmetric.

3.3. Third kind of curvature tensor

For curvature tensor of the third kind (2.13) of nonsymmetric connection we have $u = 1,\ u' = 1,\ v = -1,\ v' = 1,$ $w = -2$ and by replacing in Equations (3.4) and (3.5) we see that $\underset{3}{\beta}_{ij} = \underset{1}{\beta}_{ij}$ and $\underset{3}{\gamma}_{ij} = \underset{2}{\beta}_{ij}$. Consequently, we have

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} = R^i_{jmn} &+ \delta^i_j (\underset{1}{\beta}_{mn} + \underset{2}{\beta}_{nm}) - \delta^i_m \underset{1}{\beta}_{jn} - \delta^i_n \underset{2}{\beta}_{jm} \\ &+ F^i_j (F^p_m \underset{1}{\beta}_{pn} + F^p_n \underset{2}{\beta}_{pm}) - F^i_m F^p_j \underset{1}{\beta}_{pn} - F^i_n F^p_j \underset{2}{\beta}_{pm}.\end{aligned} \tag{3.25}$$

By introducing the tensors

$$\underset{12}{\alpha}_{ij} = \underset{1}{\beta}_{ij} + \underset{2}{\beta}_{ji}, \;\; \underset{12}{\varepsilon}_{ij} = F^p_i \underset{1}{\beta}_{pj} + F^p_j \underset{2}{\beta}_{pi} \tag{3.26}$$

the curvature tensor $\underset{3}{\overline{R}}{}^i_{jmn}$ of product semisymmetric connection takes the following form

$$\underset{3}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + \delta^i_j \underset{12}{\alpha}_{mn} - \delta^i_m \underset{1}{\beta}_{jn} - \delta^i_n \underset{2}{\beta}_{jm} + F^i_j \underset{12}{\varepsilon}_{mn} - F^i_m \underset{1}{\zeta}_{jn} - F^i_n \underset{2}{\zeta}_{jm}. \tag{3.27}$$

The tensors $\underset{12}{\alpha}_{ij}$ and $\underset{12}{\varepsilon}_{ij}$ are symmetric.

3.4. Fourth kind of curvature tensor

Now, let us consider the curvature tensor of fourth kind of nonsymmetric connection. From Equation (2.13) we have $u=1,\ u'=1,\ v=-1,\ v'=1,\ w=2$ and after substituting these parameters in Equations (3.4) and (3.5), in the same way as before, we get the curvature tensor of the fourth kind of product semisymmetric connection in a locally decomposable Riemannian space

$$\begin{aligned}\underset{4}{\overline{R}}{}^i_{jmn} &= R^i_{jmn} + \delta^i_j(\underset{4}{\beta}_{mn} + \underset{4}{\gamma}_{nm}) - \delta^i_m \underset{4}{\beta}_{jn} - \delta^i_n \underset{4}{\gamma}_{jm} \\ &\quad + F^i_j(F^p_m \underset{4}{\beta}_{pn} + F^p_n \underset{4}{\gamma}_{pm}) - F^i_m F^p_j \underset{4}{\beta}_{pn} - F^i_n F^p_j \underset{4}{\gamma}_{pm},\end{aligned} \tag{3.28}$$

i.e.

$$\underset{4}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + \delta^i_j \underset{4}{\alpha}_{mn} - \delta^i_m \underset{4}{\beta}_{jn} - \delta^i_n \underset{4}{\gamma}_{jm} + F^i_j \underset{4}{\varepsilon}_{mn} - F^i_m \underset{4}{\zeta}_{jn} - F^i_n \underset{4}{\eta}_{jm}, \tag{3.29}$$

where

$$\underset{4}{\beta}_{ij} = \tau_{i;j} - 3(\tau_i\tau_j + F^p_i F^q_j \tau_p\tau_q), \quad \underset{4}{\gamma}_{ij} = \tau_{i;j} + 3(\tau_i\tau_j + F^p_i F^q_j \tau_p\tau_q), \tag{3.30}$$

$$\underset{4}{\alpha}_{ij} = \underset{4}{\beta}_{ij} + \underset{4}{\gamma}_{ji}, \quad \underset{4}{\varepsilon}_{ij} = \underset{4}{\zeta}_{ij} + \underset{4}{\eta}_{ji}, \quad \underset{4}{\zeta}_{ij} = F^p_i \underset{4}{\beta}_{pj}, \quad \underset{4}{\eta}_{ij} = F^p_i \underset{4}{\gamma}_{pi}. \tag{3.31}$$

The tensors $\underset{4}{\alpha}_{ij}$ and $\underset{4}{\varepsilon}_{ij}$ are symmetric.

3.5. Fifth kind of curvature tensor

For curvature tensor of the fifth kind of nonsymmetric connection from Equation (2.13) we have $u=0,\ u'=0,$ $v=1,\ v'=1,\ w=0$ and after replacing in Equations (3.4) and (3.5) we obtain the curvature tensor of the fifth kind of product semisymmetric connection

$$\underset{5}{\overline{R}}{}^i_{jmn} = R^i_{jmn} + 2\delta^i_j \underset{5}{\beta}_{mn} - \delta^i_m \underset{5}{\beta}_{jn} - \delta^i_n \underset{5}{\beta}_{jm} + 2F^i_j \underset{5}{\zeta}_{mn} - F^i_m \underset{5}{\zeta}_{jn} - F^i_n \underset{5}{\zeta}_{jm}, \tag{3.32}$$

where

$$\underset{5}{\beta}_{ij} = \tau_i\tau_j + F^p_i F^q_j \tau_p\tau_q, \quad \underset{5}{\zeta}_{ij} = F^p_i \underset{5}{\beta}_{pj}. \tag{3.33}$$

The previous equation shows that the tensors $\underset{5}{\beta}_{ij}$ and $\underset{5}{\zeta}_{ij}$ are symmetric.

Now we will observe some properties of the curvature tensors in the locally decomposable Riemannian space, which are obtained with respect to the covariant derivative of product structure tensor field F^i_j.

Theorem 3.4 *Let $\underset{\theta}{\overline{R}}{}^i_{jmn}$, $\theta = 1,2,\ldots,5$ be the curvature tensor of the locally decomposable Riemannian space with respect to the product semisymmetric connection (2.5), then the relations*

$$F^p_j \underset{\theta}{\overline{R}}{}^i_{pmn} = F^i_p \underset{\theta}{\overline{R}}{}^p_{jmn}, \ \theta = 1,2,3, \tag{3.34}$$

$$\underset{4}{\overline{R}}{}^i_{jmn} + \underset{3}{\overline{R}}{}^i_{jnm} = 2\left(T^i_{jm\underset{4}{|}n} + T^i_{jn\underset{3}{|}m} + 2T^p_{jm}T^i_{pn} - 2T^p_{jn}T^i_{pm}\right), \tag{3.35}$$

are valid, where $\underset{\vartheta}{|}$ denoting the covariant derivative of ϑ-th kind, $\vartheta = 1,2,3,4$, with respect to the product semisymmetric connection $\overline{\Gamma}^i_{jk}$.

Proof For the first equation you can see Theorem 18.2 in [9]. By using covariant differentiation of the third and fourth kind and Equation (2.19), we obtain

$$F^i_{\underset{3}{j|m}\underset{4}{|n}} \overset{2.19}{=} 2\left(F^p_j T^i_{pm}\right)_{\underset{4}{|n}} = 2F^p_{\underset{4}{j|n}} T^i_{pm} + 2F^p_j T^i_{\underset{4}{pm|n}} \overset{2.19}{=} -4F^p_j T^q_{pn} T^i_{qm} + 2F^p_j T^i_{\underset{4}{pm|n}}. \tag{3.36}$$

Similarly

$$F^i_{\underset{4}{j|n}\underset{3}{|m}} = -4F^p_j T^q_{pm} T^i_{qn} - 2F^p_j T^i_{\underset{3}{pn|m}}. \tag{3.37}$$

By using the eleventh Ricci type identity (equation (56') in [5]), i.e.

$$F^i_{\underset{3}{j|m}\underset{4}{|n}} - F^i_{\underset{4}{j|n}\underset{3}{|m}} = F^p_j \underset{4}{\overline{R}}{}^i_{pmn} + F^i_p \underset{3}{\overline{R}}{}^p_{jnm} \tag{3.38}$$

and by virtue of Equations (3.36) and (3.37), we have

$$2F^p_j \left(T^i_{\underset{4}{jm|n}} + T^i_{\underset{3}{jn|m}} + 2T^p_{jm} T^i_{pn} - 2T^p_{jn} T^i_{pm} \right) = F^p_j \underset{4}{\overline{R}}{}^i_{pmn} + F^i_p \underset{3}{\overline{R}}{}^p_{jnm}. \tag{3.39}$$

Finally, considering Equation (3.34), we obtain (3.35). □

It can easily be shown that the equation (3.34) also holds for the Riemannian curvature tensor, i.e.

$$F^p_j R^i_{pmn} = F^i_p R^p_{jmn}. \tag{3.40}$$

At the end of this section, we can make a connection between $\underset{1}{\beta}_{ij}$ and other introduced tensors:

$$\underset{2}{\beta}_{ij} = 2\tau_{i;j} - \underset{1}{\beta}_{ij},\ \underset{4}{\beta}_{ij} = 4\tau_{i;j} - 3\underset{1}{\beta}_{ij},\ \underset{4}{\gamma}_{ij} = 3\underset{1}{\beta}_{ij} - 2\tau_{i;j},\ \underset{5}{\beta}_{ij} = \underset{1}{\beta}_{ij} - \tau_{i;j}. \tag{3.41}$$

4. Symmetry properties of curvature tensors

In the continuation of the paper, we will observe the symmetry properties of curvature tensors in a locally decomposable Riemannian space. By virtue of Equations (3.19) and (3.20), for the first kind of curvature tensor of product semisymmetric connection, we get

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} &= -R^i_{jnm} - \delta^i_j \underset{1}{\alpha}_{nm} - \left(-\delta^i_n \underset{1}{\beta}_{jm} + \delta^i_m \underset{1}{\beta}_{jn} \right) - F^i_j \underset{1}{\varepsilon}_{nm} - \left(-F^i_n \underset{1}{\zeta}_{jm} + F^i_m \underset{1}{\zeta}_{jn} \right) \\ &= -\left(R^i_{jnm} + \delta^i_j \underset{1}{\alpha}_{nm} - \delta^i_n \underset{1}{\beta}_{jm} + \delta^i_m \underset{1}{\beta}_{jn} + F^i_j \underset{1}{\varepsilon}_{nm} - F^i_n \underset{1}{\zeta}_{jm} + F^i_m \underset{1}{\zeta}_{jn} \right),\end{aligned}$$

and from there

$$\underset{1}{\overline{R}}{}^i_{jmn} = -\underset{1}{\overline{R}}{}^i_{jnm}, \tag{4.1}$$

where the antisymmetry property of Riemannian curvature tensor is taken into consideration. The previous equation means that the first kind of curvature tensor is antisymmetric with regard to indices m and n.

Based on equation (3.15), for tensor $\underset{1}{\overline{R}}{}^i_{jmn}$, we get

$$\underset{jmn}{Cycl}\, \underset{1}{\overline{R}}{}^i_{jmn} = 2 \underset{jmn}{Cycl} \left(\delta^i_j \tau_{[m;n]} + F^i_j F^p_m \tau_{[p;n]} \right). \tag{4.2}$$

In a similar manner, the symmetry properties for the other curvature tensors of product semisymmetric connection can be examined, so we proved the following theorem.

Theorem 4.1 *Let* $\underset{\theta}{\overline{R}}{}^i_{jmn}$, $\theta = 1, 2, \ldots, 5$, *be the curvature tensor of product semisymmetric connection (2.5) in the locally decomposable Riemannian space. The relations*

$$\underset{1}{\overline{R}}{}^i_{jmn} = -\underset{1}{\overline{R}}{}^i_{jnm}, \quad \underset{jmn}{Cycl}\, \underset{1}{\overline{R}}{}^i_{jmn} = 2 \underset{jmn}{Cycl} \left(\delta^i_j \tau_{[m;n]} + F^i_j F^p_m \tau_{[p;n]} \right), \tag{4.3}$$

$$\underset{2}{\overline{R}}{}^i_{jmn} = -\underset{2}{\overline{R}}{}^i_{jnm}, \quad \underset{jmn}{Cycl}\, \underset{2}{\overline{R}}{}^i_{jmn} = -2 \underset{jmn}{Cycl} \left(\delta^i_j \tau_{[m;n]} + F^i_j F^p_m \tau_{[p;n]} \right), \tag{4.4}$$

$$\underset{3}{\overline{R}}{}^i_{jmn} \neq \pm \underset{3}{\overline{R}}{}^i_{jnm}, \quad \underset{jmn}{Cycl}\, \underset{3}{\overline{R}}{}^i_{jmn} = 0, \tag{4.5}$$

$$\underset{4}{\overline{R}}{}^i_{jmn} \neq \pm \underset{4}{\overline{R}}{}^i_{jnm}, \quad \underset{jmn}{Cycl}\, \underset{4}{\overline{R}}{}^i_{jmn} = 0, \tag{4.6}$$

$$\underset{5}{\overline{R}}{}^i_{jmn} \neq \pm \underset{5}{\overline{R}}{}^i_{jnm}, \quad \underset{jmn}{Cycl}\, \underset{5}{\overline{R}}{}^i_{jmn} = 0, \tag{4.7}$$

are valid.

Based on the previous theorem, we have direct consequences.

Corollary 4.2 *The curvature tensors* $\underset{1}{\overline{R}}{}^i_{jmn}$ *and* $\underset{2}{\overline{R}}{}^i_{jmn}$ *are antisymmetric with respect to the indices* m, n, *while the curvature tensors* $\underset{3}{\overline{R}}{}^i_{jmn}$, $\underset{4}{\overline{R}}{}^i_{jmn}$ *and* $\underset{5}{\overline{R}}{}^i_{jmn}$ *are not antisymmetric.*

Corollary 4.3 *The curvature tensors* $\underset{1}{\overline{R}}{}^i_{jmn}$ *and* $\underset{2}{\overline{R}}{}^i_{jmn}$ *are not cyclically symmetric with respect to the indices* j, m, n, *while the curvature tensors* $\underset{3}{\overline{R}}{}^i_{jmn}$, $\underset{4}{\overline{R}}{}^i_{jmn}$ *and* $\underset{5}{\overline{R}}{}^i_{jmn}$ *are cyclically symmetric.*

5. Special case of vector τ_i

Depending on the vector τ_i and its properties, special cases of product semisymmetric connection can be observed. We will now look at the case where τ_i is a gradient vector and see what happens to the curvature tensor properties.

Theorem 5.1 *Let* $(\mathcal{M}_N, g, F)$ *be a locally decomposable Riemannian space endowed with the product semisymmetric connection (2.5). If* τ_i *is a gradient vector field then the curvature tensor* $\overline{R}^i_{jmn}$ *is cyclic-symmetric.*

Proof As for the gradient vector applies that $\tau_{i;j} = \tau_{j;i}$, from here it follows that $\tau_{[i;j]} = 0$, so based on (3.15) we directly conclude that the claim for cyclic symmetry is valid. □

On the basis of Equation (3.15) it is easy to conclude that curvature tensors $\overline{R}^i_{jmn}$ will be cyclically symmetric also in the case that $u = u'$.

Theorem 5.2 *Let $(\mathcal{M}_N, g, F)$ be a locally decomposable Riemannian space endowed with the product semisymmetric connection (2.5).*

1. *The Ricci tensor $\overline{R}_{jm} = \overline{R}^p_{jmp}$ is symmetric, and*

2. *the covariant curvature tensor $\overline{R}_{ijmn} = g_{ip}\overline{R}^p_{jmn}$ is invariant under changing places of the pairs of indices ij and mn, i.e. $\overline{R}_{ijmn} = \overline{R}_{mnij}$,*

if and only if τ_i is a gradient vector field.

Proof

1. By contracting with respect to the indices i and n in Equation (3.7) we obtain

$$\overline{R}_{jm} = R_{jm} + \alpha_{mj} - \beta_{jm} - N\gamma_{jm} + F^p_j \varepsilon_{mp} - F^p_m \zeta_{jp} - \varphi\eta_{jm}, \tag{5.1}$$

where $\varphi = N_1 - N_2$. Using the equations from Theorem 3.1, we can easily conlclude that $F^p_j\varepsilon_{ip} = \alpha_{ij}$ and $F^p_j\zeta_{ip} = \beta_{ij}$, so the previous equation takes the form

$$\overline{R}_{jm} = R_{jm} + 2\alpha_{mj} - 2\beta_{jm} - N\gamma_{jm} - \varphi\eta_{jm}. \tag{5.2}$$

Substituting expressions for tensors α_{ij} and β_{ij}, we obtain the form of Ricci tensor of product semisymmetric connection in the locally decomposable Riemannian space

$$\overline{R}_{jm} = R_{jm} - 2u\tau_{[j;m]} + (2-N)\gamma_{jm} + -\varphi\eta_{jm}. \tag{5.3}$$

Interchanging j and m in this equation, we obtain

$$\overline{R}_{mj} = R_{mj} - 2u\tau_{[m;j]} + (2-N)\gamma_{mj} + -\varphi\eta_{mj}, \tag{5.4}$$

wherefrom, after rearanging, one obtains

$$\overline{R}_{jm} - \overline{R}_{mj} = \left((2-N)\,u' - 4u\right)\tau_{[j;m]} - \varphi u' F^p_j \tau_{[p;m]}. \tag{5.5}$$

Based on the last equation, we can easily conclude that Ricci tensor is symmetric, i.e. $\overline{R}_{jm} = \overline{R}_{mj}$, if and only if τ_i is a gradient vector.

2. By lowering the index i in the curvature tensor $\overline{R}^i_{jmn}$ we get the form of covariant curvature tensor

$$\overline{R}_{ijmn} = R_{ijmn} + g_{ij}\alpha_{mn} - g_{im}\beta_{jn} - g_{in}\gamma_{jm} + F_{ij}\varepsilon_{mn} - F_{im}\zeta_{jn} - F_{in}\eta_{jm}. \tag{5.6}$$

If we assume that $\overline{R}_{ijmn} = \overline{R}_{mnij}$, we come to the equation

$$\begin{aligned} &g_{ij}\alpha_{mn} - g_{mn}\alpha_{ij} - g_{im}\beta_{jn} + g_{mi}\beta_{nj} - g_{in}\gamma_{jm} + g_{mj}\gamma_{ni} \\ &+ F_{ij}\varepsilon_{mn} - F_{mn}\varepsilon_{ij} - F_{im}\zeta_{jn} + F_{mi}\zeta_{nj} - F_{in}\eta_{jm} + F_{mj}\eta_{ni} = 0. \end{aligned} \tag{5.7}$$

where it has been taken into account that for the Riemannian tensor applies $R_{ijmn} = R_{mnij}$. By composing the previous equation with g^{im} and using the simple calculus as in the previous part of the theorem, we have

$$\left((2-N)\,u' - 4u\right)\tau_{[j;m]} - \varphi u' F^p_j \tau_{[p;m]} = 0, \tag{5.8}$$

so we conclude that equation $\overline{R}_{ijmn} = \overline{R}_{mnij}$ is valid if and only if τ_i is a gradient vector, which proves the theoreme. □

Based on Equations (5.5) and (5.8) it is easy to conclude that $\overline{R}_{jm} = \overline{R}_{mj}$ and $\overline{R}_{ijmn} = \overline{R}_{mnij}$ in the case that $u = u' = 0$.

The symmetry of the Ricci tensor indicates that equiaffine space, that is, when τ_i is a gradient vector, then locally decomposable Riemannian space is equaffine space with respect to the product semisymmetric connection.

In case when τ_i is a gradient vector, then expressions for curvature tensors $\underset{1}{\overline{R}}{}^i_{jmn}$ and $\underset{2}{\overline{R}}{}^i_{jmn}$ have simpler form, i.e. the tensors $\underset{1}{\alpha}_{ij}$, $\underset{2}{\alpha}_{ij}$, $\underset{1}{\varepsilon}_{ij}$, $\underset{2}{\varepsilon}_{ij}$ vanish.

6. Conclusion

At the beginning of the paper, we have given some relations that satisfy the almost product structure tensor and torsion tensor of product semisymmetric connection. We proved that the value of the Nijenhuis tensor in a locally decomposable Riemannian space is zero, i.e. Nijenhuis tensor vanishes.

We found the generalized form of the curvature tensor $\overline{R}^i_{jmn}$ for product semisymmetric connection in a locally decomposable Riemannian space, and then we stated five linearly independent tensors $\underset{\theta}{\overline{R}}{}^i_{jmn}$, $\theta = 1, 2, \ldots, 5$. It is interesting that tensors $\underset{\theta}{\overline{R}}{}^i_{jmn}$, $\theta = 1, 2, \ldots, 5$, which are antisymmetric by index pair m, n are not cyclically symmetric by j, m, n, and vice versa, which are not antisymmetric by m, n are cyclically symmetric by j, m, n.

Finally, we observed the special case where the generator of product semisymmetric connection is a gradient vector and we have shown that in this case, it is possible to generalize the properties for the curvature tensor of this connection, that is, the gradient vector τ_i implies cyclic symmetry of the tensor $\overline{R}^i_{jmn}$, symmetry of the Ricci tensor $\overline{R}_{jm}$, as well as invariance of the covariant curvature tensor $\overline{R}_{ijmn}$ under interchanging index pair ij with mn.

Acknowledgment

The authors were supported by the research projects 174025 and 174012 of the Ministry of Education, Science and Technological Development, Republic of Serbia. (451-03-68/2020-14/200123)

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