APPLICATION OF THE FOKKER-PLANCK-KOLMOGOROV EQUATION TO DETERMINING THE MEAN VALUE OF NUCLEONS GENERATED BY THE DIRECT EJECTION PROCESS

by

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The statistical description of the process of direct nucleon ejection is the subject of this paper. This description is based on the generalized Fokker-Planck-Kolmogorov equation. The basic proposal is this: deterministic equations and their solutions have the mean values of the stochastic model of the ablation problem. The problem of deformation of the phase transition front is considered. The study is carried out by using the introduced stability position for the dispersion of solutions for mean values. The result of the study is the conclusion that the influence of the Markov diffusion coefficient leads to distortion of the original shape of the boundary phase transition front. The effect of the initial aspiration to resist changing the shape of the phase transition front was found.

Key words: Stefan problem, ejection problem, Fokker-Planck-Kolmogorov equation, differential equation, phase transition front

INTRODUCTION

The problem of ablation has many practical implementations [1, 2]. One of them is the application to the direct ejection process, based on the interaction of high energy particles whose associated wavelengths are smaller than the nucleus so they interrelate directly with the nucleons. Besides nuclear physics, there are also applications in other areas. Firstly, it is the process of transferring the substance from the surface of a solid body under the action of radiation and the hot gas flow. Secondly, it is the reduction of the glacial mass or snow cover as a result of melting and evaporation, depending largely on climatic factors. Thirdly, it is the removal (evaporation) of a substance from the surface under the action of laser radiation. The description of the ablation has the type of Stefan problem of thermal conductivity during phase transformations [3].

It is proposed here to consider such a problem in a stochastic view. In this direction, the first results are presented in publications [4-7]. The basic proposition is the following: deterministic equations and their solutions are the mean values of a stochastic model. The main attention, which was dedicated to Stefan problem [8], was made to explain the mechanisms of the appearance of instability of the shape of change of the phase transition front.

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Based on the generalized equation of Fokker-Planck-Kolmogorov (FPK) for the probability density (PD) [9], the equation for temperature dispersion was derived and conditions of stability by dispersion were introduced. Let us start by specifying the definitions of stability in the stochastic sense by dispersion of the solution of the problem for the mean value [9].

Definition 1. We call the solution of a medium-value problem stable in the stochastic sense by dispersion, if for any $\varepsilon > 0$ is possible to choose such $\delta(\varepsilon) > 0$ that for the initial value of the dispersion $D_{T_0} \quad \delta(\varepsilon)$ dispersion $D_{\mathrm{T}}(t)$ for all $t > t_0$ satisfied inequalities $D_T(t) < \varepsilon$.

Definition 2. We call the solution of the problem for the mean value asymptotically stable in the stochastic sense by dispersion, if for the initial value of the dispersion $D_{T_0} \quad \delta(\varepsilon)$, where the dispersion $D_T(t)$ satisfies the condition $\lim_{t \quad \infty} D(t) \quad G \quad \text{const} \quad \infty$.

For $D_{T_0} \quad 0$ and $G = 0$, the solution will be called absolutely stable in the stochastic sense by dispersion. Note that for Markov zero diffusion coefficient, stability conditions in the stochastic sense turn to the classical Lyapunov stability conditions [10].

As a result of the research of the classical Stefan problem [8], *strange* dispersion behavior was identified at the initial moment. The essence of the effect is the following: the regular component of the dispersion from its initial value decreases its initial value to some moment of time $t_{\min}$ and reaches a minimum. This means that at this time interval there is a resistance to the change of shape of the phase transition front. After a certain moment $t_{\min}$ this resistance weakens, the dispersion starts to grow and the phase of active deformation of the shape of the phase transition front begins. This seemingly small effect, due to the lowness of the coefficient of Markov's diffusion and time, can be important for fine technological processes, when during the process it is required to preserve longer the original flat form, or raised crystal, or a flat shape of a solid body, with other thermal action, which does not have to possess the character of phase transformations, for example, when nano coatings are applied.

Knowing the dispersion change in time, it is necessary to run the technological process gradually starting from $t = 0$ to $t_{\min}$, then it is necessary to stop, and start again and periodically repeat this procedure many times. This will level out unnecessary random effects that contribute to the deformation of a flat shape of a solid body subject, to external and internal accidental influences.

THE DERIVATION OF THE GENERALIZED FOKKER-PLANCK-KOLMOGOROV EQUATION

Why did we have to derive the equation for the probability density, which we called a generalized? The fact is that the classical equation includes only a time coordinate and there are no spatial co-ordinates. If the classical equation allows one to obtain only the Cauchy problem for an ordinary differential equation with respect to the mean values proposed in [5], a generalization makes it possible to construct an initial-boundary value problem that corresponds to partial differential equations. To clarify further arguments, we reproduce the derivation of the generalized FPK.

The basis for obtaining stochastic analogues of problems corresponding to deterministic problems is the representation we proposed in [8] that the solution of the deterministic problem completely coincides with the solution of the problem for mean values. Let us mark the probability density function (hereinafter PD) with $P(t,x,\Omega)$. The classical FPK equation, written without the term responsible for the discontinuity, has the following form

$$\frac{\partial P(t,x,\Omega)}{\partial t} = -\frac{\partial[A(t,x,\Omega)P(t,x,\Omega)]}{\partial \Omega} + 0.5B_{\Omega}\frac{\partial^2 P(t,x,\Omega)}{\partial \Omega^2} \tag{1}$$

Here, $A(t, x, \Omega)$ is the drift coefficient, B – the Markov diffusion coefficient of a random phenomenon, t – the time, x – a space co-ordinate, and Ω – the characteristic of the random field.

We modify eq. (1) to describe the random thermal fields. The mean value of the temperature and the second moment will be denoted by

$$M_T^{(1)}(t,x) = \int_{-\infty}^{\infty} \Omega P(t,x,\Omega)\mathrm{d}\Omega$$

$$M_T^{(2)}(t,x) = \int_{-\infty}^{\infty} \Omega^2 P(t,x,\Omega)\mathrm{d}\Omega$$

Mark the dispersion with

$$D_T^2(t,x) = M_T^{(2)}(t,x) - [M_T^{(1)}(t,x)]^2,$$

the question arises as to how to set the drift coefficient, so that from the equation for PD one would obtain an equation for mean values that would coincide with the classical heat equation for deterministic temperature? The answer to this question can be obtained by an analytical solution of the following inverse problem. We represent the expansion of the drift coefficient in the Maclaurin series in order of Ω

$$A(t,x,\Omega) = \sum_{k=1}^{\infty} \alpha_k(t,x)\Omega^k$$

It is no coincidence that the series starts with the member $a_1(t, x)\Omega$, since using a coefficient $\alpha_1(t, x)$, which does not depend on Ω would lead to a deterministic Liouville phenomenon with zero dispersion.

The equation for the mean values of the form

$$\frac{\partial M_T^{(1)}(t,x)}{\partial t} = a\frac{\partial^2 M_T^{(1)}(t,x)}{\partial x^2} \tag{2}$$

could be obtained from eq. (1), where a is the temperature conduction coefficient, if the necessary and sufficient conditions are satisfied

$$a_1(t,x) = \frac{a\dfrac{\partial^2}{\partial x^2}M_T^{(1)}(t,x)}{M_T^{(1)}(t,x)}, \alpha_k(t,x) = 0, k = 2,3,\ldots n$$

Indeed, by introducing $A(t, x, \Omega)$ in eq. (1)

$$\frac{\partial P(t,x,\Omega)}{\partial t} = -\frac{\partial}{\partial \Omega}\left[\frac{a\dfrac{\partial^2}{\partial x^2}M_{\Omega}^{(1)}(t,x)}{M_{\Omega}^{(1)}(t,x)}\Omega P(t,x,\Omega)\right] + 0.5B_{\Omega}\frac{\partial^2 P(t,x,\Omega)}{\partial \Omega^2} \tag{3}$$

multiplying both sides of eq. (3) by Ω and integrating over this variable in infinite boundaries, we obtain the following equation

$$\int_{-\infty}^{\infty} \Omega \frac{\partial P(t,x,\Omega)}{\partial t} \mathrm{d}\Omega = -\int_{-\infty}^{\infty} \Omega \frac{\partial \left(\dfrac{a \dfrac{\partial}{\partial x^2} M_\Omega^{(1)}}{M_\Omega^{(1)}} \Omega P(t,x,\Omega) \right)}{\partial \Omega} \mathrm{d}\Omega + 0.5 B_\Omega \int_{-\infty}^{\infty} \Omega \frac{\partial^2 P(t,x,\Omega)}{\partial \Omega^2} \mathrm{d}\Omega \tag{4}$$

We rearrange the operations of integration and differentiation on the left-hand side of the eq. (4), and on the right-hand side we integrate by parts. Then, taking into account the equality of the PD and its derivatives at infinity, we obtain an equation for the mean values, which coincide with the classical heat eq. (2).

If initial and boundary conditions of the form

$$P(0,x,\Omega) = P_{\mathrm{init}}(x,\Omega), x \in [0,l], P(t,0,\Omega) = P_0(t,\Omega)$$
$$P(t,l,\Omega) = P_1(t,\Omega), t \in [0, \infty), \Omega \in (-\infty, +\infty)$$

are added to eq. (3), then we get the formulation of the problem for the mean values of the form eq. (2).

The problem for dispersion at the boundary of the phase transition has the following form

$$\frac{\mathrm{d}D_T(t)}{\mathrm{d}t} = \frac{1}{L\rho M_T^{(1)}(t)} \left[\lambda_1 \frac{\partial M_T^{(1)}(t,x)}{\partial x}\bigg|_{x=M_T^{(1)}(t)} - \lambda_2 \frac{\partial^2 M_2^{(1)}(t,x)}{\partial x^2}\bigg|_{x=M_T^{(1)}(t)} \right] D_{\mathrm{T}}(t) + B_\Omega, t > 0 \tag{5}$$

$$D_{\mathrm{T}}(0) = D_{T_{\mathrm{init}}} = \delta_{T_0}^2 (M_{T_0}^{(1)})^2 \tag{6}$$

where ρ is the density and L – the heat of phase transition.

Equation (5) can be expressed as

$$\frac{\mathrm{d}D_{\mathrm{T}}(t)}{\mathrm{d}t} = \frac{\dfrac{d}{\mathrm{d}t} M_T^{(1)}(t)}{M_T^{(1)}(t)} D_{\mathrm{T}}(t) + B_\Omega \tag{7}$$

and solution of problem (6)-(7) has a form

$$D_{\mathrm{T}}(t) = \left[\int_0^t \frac{B_\Omega \mathrm{d}\xi}{(M_T^{(1)}(\xi))^2} + \delta_0^2 \right] (M_T^{(1)}(t))^2, \quad t \in [0, \infty) \tag{8}$$

INVESTIGATION OF THE DEFORMATION OF THE PHASE TRANSITION FRONT

The fact that different origin front shape deformation is often observed, caused by anthropogenic, natural and technological influences, is explained by the random character of all these phenomena. It has been shown that the special role in description of these phenomena is given by the diffusion coefficient of Markov field, which primarily determines the distortion of the front.

It was established that the size of the deformation zone of the phase transition front, during the ablation, increases proportionally with $\sqrt{t}$ [1, 2]. If the speed of the front movements is less than, or equal to the growth rate of the deformation zone, then it is necessary to speak of the instability of this front. It means the front itself does not exist. A similar presentation is determined by statistical distribution and development during the time of centers of the new phase formation, which also determine the size of Markov diffusion coefficient.

There are different approaches to explaining the occurrence and mechanisms of instability of the form of the transition phase front. In most cases the description of instability is based on the study of the deterministic causes of the *deformation* of the phase transition boundary. In fact, considering that there is a positive gradient in some part of this boundary, a small forward displacement of a certain part can lead to an increase in the heat flux and the speed of occurrence of a new phase, therefore, so that a *protrusion* of the front takes place on this section. In this part there is a *convexity* of the front. The curvature of the surface of the phase transition is of great importance: the convex sections grow faster than the concave ones.

However, this deterministic explanation does not take into account the reasons for the random distribution of the centers of a new phase formation, which can be described only within a stochastic model. The integration of stochastic concepts of the phase formation kinetics, in combination with the models proposed here, allows one to estimate, on one hand, the diffusion coefficient of the Markov random field, and, on the other hand, its connection with the stability conditions of the phase transition front. From the solution (8), according to the law of motion of the front in the form of $M_T^{(1)}(t) = \alpha\sqrt{t}$, it follows that the dispersion for $\delta_0^2 = 0, t \in [0, \infty)$ has a form

$$D_{\mathrm{T}}(t) = \left[\int_0^t \frac{B_\Omega \mathrm{d}\xi}{(\alpha\sqrt{\xi})^2} + \delta_0^2 \right] (\alpha\sqrt{t})^2 = B_\Omega t \ln t$$

We denote by *Reg D* (t) the value of the product of the improper integral by the mean value of the law of motion of the front

$$Reg\, D(t) = \alpha t \int_0^t \frac{B_\Omega \mathrm{d}\xi}{(\alpha\sqrt{\xi})^2} = B_\Omega t \ln t$$

where *Reg D*(t) is regular dependence dispersion of time.

The minimum *Reg D*(t) is attained at $t_{\min} = 1/e$ and has a value

$$|Reg\, D(t_{\min})| = \left| -\frac{B_\Omega}{e} \right| = \frac{B_\Omega}{e}$$

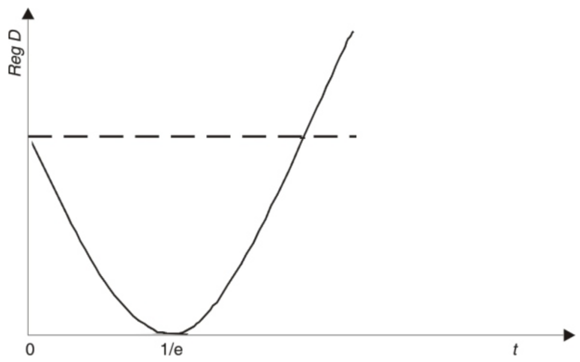

Figure 1. Dispersion dependence on time, which illustrates the effect of resistance to change of the phase transition front at the initial moments of time

The dependency graph $Reg\,D$ is shown in fig. 1.

It turns out that at the initial moments of time the dispersion decreases and the initial configuration of the front resists its original state.

From this conclusion, it follows that it is necessary to build a controlled process of motion by the front so as to stop at the moment of minimal dispersion, then the original front surface will be close to the original one. If this is not taken into account, then a strong distortion will result, such as, for example, as we observe with icicles on roofs of buildings.

CONCLUSIONS

The obtained result is important and applicable to the processes of direct ejection in high energy physics. In cases where the process of movement of the front is impossible to manage, when the phenomenon has a natural character that is still beyond the power of man, for example, weather determines the growth of icebergs or evaporation from a large surface, or the meteorite burning in the atmosphere (by the way, all these are Stefan's problems). But, on the other hand, there are many phenomena, over which a person has power and the researcher can control these phenomena.

If you manage an object so as to influence impulsively, for example, the dynamics of entering satellites into dense layers of the atmosphere, it is possible to prepare a low impact to the surface of the satellite when it lands. The landing of a controlled space object in accordance with the desire to preserve the shape of the phase transition front must be carried out by gradually slowing or gradually changing the path of the body movement. But these are the problems of the constructors. Here we can only offer to pay attention to our proposals.

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AUTHORS' CONTRIBUTIONS

Both authors have contributed to model analysis and results, theoretical analysis and literature research. The manuscript was prepared and written by D. Ć. Dolićanin-Djekić.

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ПРИМЕНА ЈЕДНАЧИНЕ ФОКЕР-ПЛАНК-КОЛМОГОРОВА ЗА ОДРЕЂИВАЊЕ СРЕДЊЕ ВРЕДНОСТИ НУКЛЕОНА НАСТАЛИХ ПРОЦЕСОМ ДИРЕКТНОГ ИЗБИЈАЊА

Тема овог рада је статистички опис процеса директног избијања нуклеона који је заснован на генерализованој Фокер-Планк-Колмогоровој једначини. Детерминистичка једначина и њена решења представљају средњу вредност стохастичког модела проблема аблације. Размотрен је проблем деформације фазног прелаза. Истраживање је спроведено коришћењем уведеног положаја стабилности за дисперзију средње вредности решења и закључак је да утицај Марковог коефицијента дифузије доводи до деформације почетног облика на граници фазног прелаза. Пронађен је ефекат којим се спречава мењање облика фронта фазног прелаза.

Кључне речи: Стефанов проблем, директно избијање, једначина Фокер-Планк-Колмогорова, диференцијална једначина, фронт фазног прелаза