Theoretical Study of Tool Holder Self-excited Oscillation in Turning Processes Using a Nonlinear Model

Branko Pejović

Faculty of Technical Sciences, Kneza Miloša 7, 38220 Kosovska Mitrovica, Serbia.

Pavel Kovač and Vladimir Pucovsky

University of Novi Sad, Faculty of Technical Sciences, Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia.

Aleksandar Todić

Faculty of Technical Sciences, Kneza Miloša 7, 38220 Kosovska Mitrovica, Serbia.

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In this paper, by observing a system which is composed from a workpiece and a tool holder, a dynamic and a mathematical nonlinear model is acquired. These models can be used as a theoretical foundation for research of self-excited oscillations, which are the object of research in this paper. All relevant oscillator factors are taken into consideration including a frictional force between flank surface and a machined new surface, which is dependent on relative system speed. For obtaining more reliable results, the characteristic friction function is expanded into the Taylor series with an arbitrary number of members regarding required accuracy. The main nonlinear differential equation of the system is solved by the method of slowly varying coefficients, which is elaborated on in detail here. One assumption is made, which states that the system has a weak nonlinearity, and respectively small damping factor. After obtaining the law of motion with relation to a larger number of influential factors, the amplitude of self-excited oscillation is determined in two different ways. Previously, this is conducted for two characteristic phases—for stationary and nonstationary modes. At the end of the paper, an analytic determination and occurrence condition of self-excited oscillations is established. This is an important factor for practical use. This is also the stability condition. The starting point for this determination was a type of experimental friction function. Derived relationships allow detailed quantitative analysis of certain parameters' influence, determination of stability, and give a reliable description of the process, which is not the case with the existing linear model. After the theoretical analysis of obtained results, a possibility for application of the suggested method in the machine tool area is presented. The derived general model based on the method of slowly varying coefficients can be directly applied in all cases where nonlinearity is not too large, which is usually the case in the field of machine tools. The greater damping factor causes a smaller amplitude of self-excited steady oscillation. Characteristics of the self-excited oscillations in the described model mostly depend on the character of the friction force. Angular frequency in mentioned nonlinear oscillations depends on the amplitude and initial conditions of movement, which is not the case in free oscillations.

1. INTRODUCTION

Oscillations are an undesirable occurrence in the area of machine tools. They reduce the precision and quality of machined surfaces and also negatively influence the productivity of the machine. The problem of oscillation is also one of the key questions in cutting machining.[1] In general, resonant vibrations are the main cause of any structure failure.[2]

Machine tools can experience two types of oscillations: forced oscillations and self-excited oscillation. More important is self-excited oscillations, in which the source of oscillations is the cutting process alone. These oscillations are characterised by the unstable operation of a machine. A simple change of machining parameters that usually leads to an occurrence of unstable operation can be avoided. The search for the source for these self-excited oscillations and finding a way for their elimination is some of the most important research in this area.[3] According to one of the theories, self-excited oscillations are generated under the influence of variable friction at the rake and flank surface of the tool.[1] The theory is also developed, based on a regenerative effect, which states that these oscillations are caused by varying cross-sections of a chip. In both these cases, the construction of a machine is not taken under consideration and only the cutting process is observed.[4,5] In recent years, intensive research has been carried out in the field of self-excited oscillations in machine tools. A whole set of assumptions, analyses, and theories about self-excited oscillations has been formed. Likewise, whole new models about their occurrence are being developed. One of the most widely applied models has originated from an analysis of a cutting process on a tool's flank surface.[1] Some authors have come to a compliant model, which predicts a larger stability area when results are compared with those from the single degree-of-freedom (SDOF) model. This model takes into account dynamic characteristics of both the workpiece and the cutting tool to attain a better approach to the physical phenomenon.[6] It is found that bifurcation control can also be used to expand the unconditionally stable region at the expense of the conditionally stable region. By softening cutting nonlinearities, a subcritical instability is achieved whereas hardening cutting nonlinearities leads to local supercritical instability.[7] Hopf bifurcation may in fact become supercritical for some parameter values if a state-dependent regenerative delay is incorporated into the model.[8] In this case, small amplitude stable periodic orbits coexist with the stable cutting solution above the stability lobes, and no periodic orbits coexist with the stable stationary cutting. The simple model of a cutting process with one degree of freedom can be also used to explore the mechanism of the chatter vibration.[9] Zhehe Yao et al. studied the effect of parametric excitation on self-excited vibration based on a model of a van der Pol-Mathieu-Duffing oscillator with a time delay.[10] It reveals that there can be a zero solution for the oscillator under the effect of parametric excitation, while it is impossible to have a stable zero equation without parametric excitation. Another simple model developed by Dombovari et al. is especially dedicated for the orthogonal cutting process and can demonstrate chattering behaviour.[11] This model is formulated as a delay differential algebraic equation (DDAE) and includes the regenerative effect of the turning process and the non-smoothness when contact between the cutting tool and the workpiece is lost. Pankaj Wahi and Anindya Chatterjee proposed a new approach to study the global dynamics of regenerative metal cutting in turning.[12] In this case, the cut surface is modeled using a partial differential equation (PDE) coupled via boundary conditions to an ordinary differential equation (ODE) modeling the dynamics of the cutting tool. Rusinek et al. have studied a nonlinear, externally forced Duffing oscillator both analytically and numerically.[13] With the use of the analytical method, stability lobes for a linear oscillator with time delay is determined and the fundamental resonance of the Duffing oscillator with time delay is calculated by means of the multiple scale method. On the other hand, there are papers which use different approaches to solving the tool vibration laws of motion. Three-dimensional simulation is known as a modern tool for solving problems in the industry.[14] Artificial neural networks are also presented as a powerful tool for accessing problems in this area of machining.[15,16] In existing literature, the most realistic model has yet not been developed to its final form. This is partly because of the unbridgeable mathematical problems which arise while solving nonlinear differential equations.

2. DYNAMIC MODEL OF A CUTTING PROCESS

From the whole dynamic system of a machine tool, the workpiece and tool holder are singled out according to Fig. 1.[1,5] The workpiece has been moving rotationally with speed $V_o = const$. The tool holder has the mass m, stiffness c, and damping b and has the ability to oscillate. The tool and workpiece are in contact through the rake and flank tool surface. Friction occurs between the tool's flank surface and the machined surface of the workpiece. Radial force F_2 occurs in the direction normal to the flank surface. Relative speed can be calculated through the expression $V = V_o \pm \dot{y}$, taking into consideration that $\dot{y} \ll V_o$. Element $\dot{y}$ presents an absolute speed. Absolute speed is assumed to be positive with sign minus in front. This allows us to keep our analysis in the range of dry friction force dependency from speed (Coulomb law of friction). Analysis made this way is mathematically more accurate than analysis made with a plus sign. In the case of plus sign use, it would not have any influence on the final result since it is oscillatory movement.

If the origin of an OY axis is in static balance, the dynamic equation of a system will be

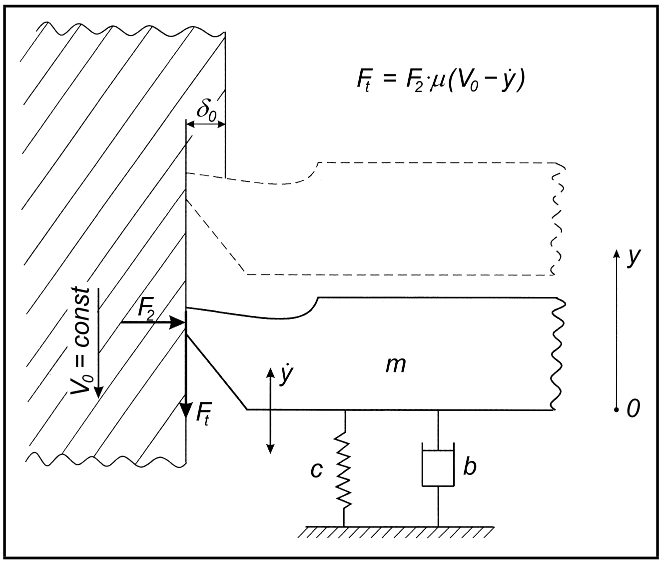

Figure 1. Model of self-excited oscillations in cutting process.

$$m \cdot \ddot{y} + b \cdot \dot{y} + c \cdot y = F_t. \tag{1}$$

Friction force is dependent on normal force F_2 and coefficient of friction:

$$F_t = \mu(V_o - \dot{y}) \cdot F_2; \tag{2}$$

where a friction function is

$$\mu(V) = \mu(V_o - \dot{y}). \tag{3}$$

Therefore a friction coefficient is dependent on relative speed $V = V_o - \dot{y}$. Substituting the Eq. (2) into Eq. (1) results in

$$m \cdot \ddot{y} + b \cdot \dot{y} + c \cdot y = \mu(V_o - \dot{y}) \cdot F_2. \tag{4}$$

Differential function $f(x)$, in area of a point x_o can be expanded into Taylor series according to an expression:

$$f(x) = f(x_o) + f'(x_o) \cdot \frac{x - x_o}{1!} + f''(x_o) \cdot \frac{(x - x_o)^2}{2!} + f'''(x_o) \cdot \frac{(x - x_o)^3}{3!} + \ldots + f^{(n)}(x_o) \cdot \frac{(x - x_o)^n}{n!} + R_n(x); \tag{5}$$

where $R_n(x)$ is the leftover of Taylor series which can be made negligibly small.[16] Based on Eq. (5) a function $f(x)$ can be derived, which is in the form of polynomial degree n, Fig. 2:[17]

$$f(x_o + h) = f(x_o) + f'(x_o) \cdot h + \frac{f''(x_o)}{2!} \cdot h^2 + \frac{f'''(x_o)}{3!} \cdot h^3 + \ldots + \frac{f^{(n)}(x_o)}{n!} \cdot h^n; \tag{6}$$

here is $h > 0$, arbitrarily small value. Based on Eq. (6), the following formula can be derived:

$$f(x_o - h) = f(x_o) - f'(x_o) \cdot h + \frac{f''(x_o)}{2!} \cdot h^2 - \frac{f'''(x_o)}{3!} \cdot h^3 + \ldots + (-1)^n \cdot \frac{f^{(n)}(x_o)}{n!} \cdot h^n. \tag{7}$$

Taking into consideration that V_o is considerably greater than $\dot{y}$ and according to analogy, with the use of Eq. (7), the

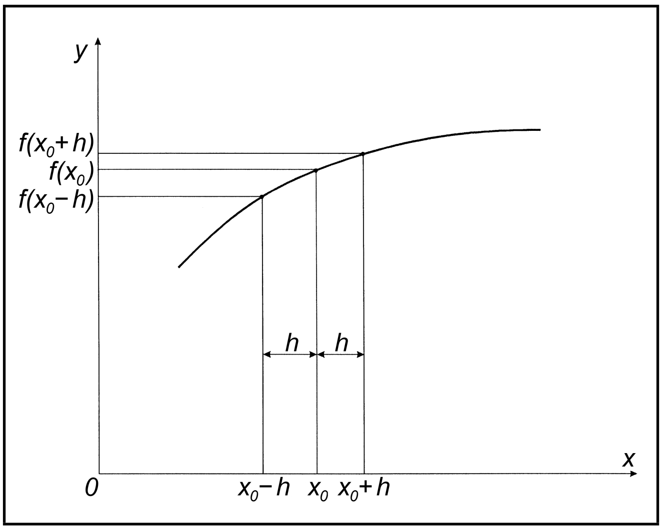

Figure 2. Development of function into Taylor series in area of a point x_o.

development of a series for friction function can be made:

$$\mu(V_o-\dot{y})=\mu(V_o)-\mu'(V_o)\cdot\dot{y}+\frac{1}{2!}\cdot\mu''(V_o)\cdot\dot{y}^2-\frac{1}{3!}\cdot\mu'''(V_o)\cdot\dot{y}^3+\ldots+(-1)^n\cdot\frac{1}{n!}\cdot\mu^{(n)}(V_o)\cdot\dot{y}^n. \quad (8)$$

Assuming that a friction function can be approximated with sufficient precision using a polynomial of the third degree ($n=3$), then it is enough to take into consideration the first four elements from Eq. (8).

Taking into consideration the prior explanation, substituting Eq. (8) into Eq. (4) will result in

$$m\cdot\ddot{y}+b\cdot\dot{y}+c\cdot y=\left[\mu(V_o)-\mu'(V_o)\cdot\dot{y}+\frac{1}{2!}\cdot\mu''(V_o)\cdot\dot{y}^2-\frac{1}{3!}\cdot\mu'''(V_o)\cdot\dot{y}^3\right]\cdot F_2. \quad (9)$$

Dividing with m, Eq. (9) can be written as

$$\ddot{y}+\omega^2\cdot y=\left[\mu(V_o)-\mu'(V_o)\cdot\dot{y}-\frac{b}{F_2}\cdot\dot{y}+\frac{1}{2!}\cdot\mu''(V_o)\cdot\dot{y}^2-\frac{1}{3!}\cdot\mu'''(V_o)\cdot\dot{y}^3\right]\cdot\frac{F_2}{m}; \quad (10)$$

where angular frequency is

$$\omega^2=\frac{c}{m}. \quad (11)$$

Equation (10) can be written as

$$\ddot{y}+\omega^2\cdot y=\left\{\mu(V_o)-\left[\mu'(V_o)+\frac{b}{F_2}\right]\cdot\dot{y}+\frac{1}{2!}\cdot\mu''(V_o)\cdot\dot{y}^2-\frac{1}{3!}\cdot\mu'''(V_o)\cdot\dot{y}^3\right\}\cdot\frac{F_2}{m}. \quad (12)$$

With an introduction of a substitution:

$$z=y-\frac{F_2}{m}\cdot\frac{\mu(V_o)}{\omega^2};\qquad \dot{z}=\dot{y};\qquad \ddot{z}=\ddot{y}; \quad (13)$$

the balanced position will be shifted, so from Eq. (12) follows:

$$\ddot{z}+\omega^2\cdot z=\frac{F_2}{m}\left[-\mu'(V_o)\cdot\dot{z}-\frac{b}{F_2}\cdot\dot{z}+\frac{1}{2!}\cdot\mu''(V_o)\cdot\dot{z}^2-\frac{1}{3!}\cdot\mu'''(V_o)\cdot\dot{z}^3\right]; \quad (14)$$

respectively:

$$\ddot{z}+\omega^2\cdot z=\left[-\frac{F_2\cdot\mu'(V_o)}{m}\cdot\dot{z}-\frac{b}{m}\cdot\dot{z}+\frac{F_2\cdot\mu''(V_o)}{2\cdot m}\cdot\dot{z}^2-\frac{F_2\cdot\mu'''(V_o)}{6\cdot m}\cdot\dot{z}^3\right]. \quad (15)$$

Damping factor b for the observed dynamic system is a small value and can be neglected,[1] thus it can be transferred in front of the brackets in Eq. (15):

$$\ddot{z}+\omega^2\cdot z=b\cdot\left\{\left[-\frac{F_2\cdot\mu'(V_o)}{b\cdot m}-\frac{1}{m}\right]\cdot\dot{z}+\frac{F_2\cdot\mu''(V_o)}{2\cdot b\cdot m}\cdot\dot{z}^2-\frac{F_2\cdot\mu'''(V_o)}{6\cdot b\cdot m}\cdot\dot{z}^3\right\}. \quad (16)$$

It is obvious that, for instance, where $b=0$, Eq. (16) becomes linear. Equation (16) can be simplified as

$$\ddot{z}+\omega^2\cdot z=b\cdot\left(-\alpha\cdot\dot{z}+\beta\cdot\dot{z}^2-\gamma\cdot\dot{z}^3\right); \quad (17)$$

where the constants are

$$\alpha=\frac{F_2\cdot\mu'(V_o)}{b\cdot m}+\frac{1}{m};\qquad \beta=\frac{F_2\cdot\mu''(V_o)}{2\cdot b\cdot m};\qquad \gamma=\frac{F_2\cdot\mu'''(V_o)}{6\cdot b\cdot m}. \quad (18)$$

3. SUGGESTION FOR PROBLEM SOLVING METHOD

One method for efficiently solving nonlinear differential Eq. (17) is presented below. This procedure is usually known as the method of slowly varying coefficients and it is realised based on the Krylov-Bogolyubov theorem.[18–21] It will be briefly explained below to give an idea of the conditions under which it is derived. In the future, this will be used for its specific application for solving an Eq. (16) and respectively Eq. (17).

In the suggested method, an assumption is made that the oscillating parameters—for example, mass—can be a variable, and its change is slow. The differential equation is a nonlinear type:

$$m(\tau)\cdot\frac{d^2x}{dt^2}+k\cdot x=\varepsilon\cdot F\left(x,\frac{dx}{dt}\right); \quad (19)$$

where $\tau=\varepsilon\cdot t$ is the so-called slowly varying coefficients and ε the small constant parameter. Function $F\left(x,\frac{dx}{dt}\right)$ is describing weak nonlinearity in the system, because of the member ε in front. Once again, the derived Eq. (17) will be transformed into Eq. (19), which is noticeable from its structure. Equation (19) can be written as

$$\frac{d^2x}{dt^2}+\omega^2(\tau)\cdot x=\varepsilon\cdot f\left(\tau,x,\frac{dx}{dt}\right); \quad (20)$$

where

$$\omega^2(\tau)=\frac{k}{m(\tau)}; \quad (21)$$

and

$$f=\frac{1}{m(\tau)}F\left(x,\frac{dx}{dt}\right). \quad (22)$$

If $\varepsilon=0$ and the mass of the system is constant ($\omega=const$), Eq. (20) will be linear and its solution would be

$$x=a\cdot\cos\psi;\qquad \frac{dx}{dt}=-a\cdot\omega\cdot\sin\psi; \quad (23)$$

where a is an amplitude and phase of oscillation:

$$\psi = \omega t + \theta. \tag{24}$$

In Eq. (24), θ is a starting phase. As stated above, the mass of the system is slowly changing and the nonlinearity is very small. In the first approximation as a rough solution, Eq. (23) can be adopted under the condition that values a and θ are considered to be dependent on the time t. This is the basic idea for solving the initial Eq. (19). Differentiation of the expression in Eq. (23) occurs twice with respect to time, taking into consideration that $a = a(t)$ and $\theta = \theta(t)$, and by using certain mathematical transformation, values of the differential $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ are determined. The next step is to integrate the obtained equation with respect to an angle ψ within the range from 0 to 2π. Using an average value for a certain range, and taking into consideration that "slow time" behaves as a constant and differential $\frac{d\psi}{dt}$ is regarded in accordance with Eq. (25), the final system of differential equations is generated:

$$\frac{da}{dt} = -\frac{\varepsilon \cdot a}{2 \cdot \omega(\tau)} \cdot \frac{d\omega(\tau)}{d\tau} - \frac{\varepsilon}{2 \cdot \pi \cdot \omega(\tau) \cdot m(\tau)} \cdot \int_0^{2\pi} F\left[a \cdot \cos\psi - a \cdot \omega(\tau) \cdot \sin\psi\right] \cdot \sin\psi \, d\psi; \tag{25}$$

$$\frac{d\psi}{dt} = \omega(\tau) - \frac{\varepsilon}{2 \cdot a \cdot \omega(\tau) \cdot m(\tau)} \cdot \int_0^{2\pi} F\left[a \cdot \cos\psi - a \cdot \omega(\tau) \cdot \sin\psi\right] \cdot \cos\psi \, d\psi. \tag{26}$$

For a specific form of function $F\left(x, \frac{dx}{dt}\right)$, in this case Eq. (17), integrals are calculated on right sides of Eq. (25) and (26). After solving differential Eqs. (25) and (26) as a system, the obtained solutions are then replaced in Eq. (23). For a particular problem according to Eq. (16):

$$b = \varepsilon; \qquad m(\tau) = 1; \tag{27}$$

$$\omega^2 = \omega^2(\tau) = const; \qquad \frac{d\omega(\tau)}{d\tau} = 0. \tag{28}$$

Function F depends only on $\frac{dz}{dt}$ in compliance with Eq. (17):

$$F = \alpha \cdot \dot{z} + \beta \cdot \dot{z}^2 + \gamma \cdot \dot{z}^3 = F\left(\frac{dz}{dt}\right). \tag{29}$$

Then the solution will be:

$$z = a(\tau) \cdot \cos\psi; \tag{30}$$

$$\frac{dz}{dt} = -a \cdot \omega \cdot \sin\psi = \dot{z}. \tag{31}$$

First element of Eq. (25), because of $\omega = const$, is equal to zero. Small value of ε is replaced by coefficient b, while element beside $\ddot{z}$ in Eq. (16) is equal to one, that is $m(\tau) = 1$. Function F is dependent only on $\dot{z} = -a \cdot \omega \cdot \sin\psi$, thus the shape of Eqs. (25) and (26) is

$$\frac{da}{dt} = -\frac{b}{2 \cdot \pi \cdot \omega} \cdot \int_0^{2\pi} F\left[-a \cdot \omega \cdot \sin\psi\right] \cdot \sin\psi \, d\psi; \tag{32}$$

$$\frac{d\psi}{dt} = \omega - \frac{b}{2 \cdot \pi \cdot a \cdot \omega} \cdot \int_0^{2\pi} F\left[-a \cdot \omega \cdot \sin\psi\right] \cdot \cos\psi \, d\psi. \tag{33}$$

Taking into consideration the form of function F, according to Eq. (29), Eqs. (32) and (33) will gain form:

$$\frac{da}{dt} = -\frac{b}{2 \cdot \pi \cdot \omega} \cdot \int_0^{2\pi} \big[-\alpha \cdot a \cdot \omega \cdot \sin\psi + \beta \cdot a^2 \cdot \omega^2 \cdot \sin^2\psi - \gamma \cdot a^3 \cdot \omega^3 \cdot \sin^3\psi\big] \cdot \sin\psi \, d\psi; \tag{34}$$

$$\frac{d\psi}{dt} = \omega - \frac{b}{2 \cdot \pi \cdot a \cdot \omega} \cdot \int_0^{2\pi} \big[-\alpha \cdot a \cdot \omega \cdot \sin\psi + \beta \cdot a^2 \cdot \omega^2 \cdot \sin^2\psi - \gamma \cdot a^3 \cdot \omega^3 \cdot \sin^3\psi\big] \cdot \cos\psi \, d\psi. \tag{35}$$

It can be noted that expressions in middle brackets under the integral are equal. Certain trigonometric integrals in Eqs. (34) and (35) will be[17]

$$\int_0^{2\pi} \sin^2\psi \, d\psi = \left[\frac{1}{2}\psi - \frac{1}{4}\sin^2\psi\right]_0^{2\pi} = \pi;$$
$$\int_0^{2\pi} \sin^3\psi \, d\psi = \left[-\cos\psi + \frac{1}{3}\cos^3\psi\right]_0^{2\pi} = 0;$$
$$\int_0^{2\pi} \sin^4\psi \, d\psi = \left[\frac{3}{8}\psi - \frac{1}{4}\sin^2\psi + \frac{1}{32}\sin^4\psi\right]_0^{2\pi} = \frac{3}{4}\pi;$$
$$\int_0^{2\pi} \sin\psi \cdot \cos\psi \, d\psi = \left[\frac{1}{2}\sin^2\psi\right]_0^{2\pi} = 0;$$
$$\int_0^{2\pi} \sin^2\psi \cdot \cos\psi \, d\psi = \left[\frac{1}{3}\sin^3\psi\right]_0^{2\pi} = 0;$$
$$\int_0^{2\pi} \sin^3\psi \cdot \cos\psi \, d\psi = \left[\frac{1}{4}\sin^4\psi\right]_0^{2\pi} = 0. \tag{36}$$

The value of a, when considering conditions for solving final equations, is considered constant. Replacing values of integral Eq. (36) into Eqs. (34) and (35) will be

$$\frac{da}{dt} = -\frac{b}{2 \cdot \pi \cdot \omega} \cdot \left(\alpha \cdot a \cdot \omega \cdot \pi + \beta \cdot a^2 \cdot \omega^2 \cdot 0 + \gamma \cdot a^3 \cdot \omega^3 \cdot \frac{3}{4}\pi\right).$$

Respectively:

$$\frac{da}{dt} = -\frac{ab}{2} \cdot \left(\alpha + \frac{3}{4}\gamma \cdot a^2 \cdot \omega^2\right). \tag{37}$$

By splitting the variables, the result of the integral is:

$$\frac{da^2}{dt} = 2a\frac{da}{dt} = -a^2 b \cdot \left(\alpha + \frac{3}{4}\gamma \cdot a^2 \cdot \omega^2\right). \tag{38}$$

Integral Eq. (38) is solved by the partial fraction method, respectively, with integration by substitution:

$$b \cdot dt = \frac{-da^2}{a^2 \cdot \left(\alpha + \frac{3}{4}\gamma \cdot a^2 \cdot \omega^2\right)}. \tag{39}$$

According to Eq. (38) and using substitution $a^2 = u$:

$$b \cdot dt = \frac{-du}{u \cdot \left(\alpha + \frac{3}{4}\gamma \cdot \omega^2 \cdot u\right)}. \tag{40}$$

Taking into consideration that:

$$\frac{1}{x(K_1 x + K_2)} = \frac{1}{K_2 x} - \frac{K_1}{K_2} \cdot \frac{1}{K_1 x + K_2};$$

respectively:

$$\int \frac{dx}{x(K_1 x + K_2)} = -\frac{1}{K_2} \cdot \ln \frac{K_1 x + K_2}{x}.$$

According to Eq. (39), it follows that:

$$b \cdot \alpha \cdot t = \ln \frac{a^2}{\frac{3}{4}\gamma\omega^2 a^2 + \alpha} + \ln C. \tag{41}$$

The integration constant is obtained from initial conditions for $t = 0$ and $a = a_0$

$$C = \frac{3}{4} \cdot \gamma \cdot \omega^2 + \frac{\alpha}{a_0^2}.$$

Replacement of a constant C in Eq. (41) will result in

$$b \cdot \alpha \cdot t = \ln \frac{\left(\frac{3}{4} \cdot \gamma \cdot \omega^2 + \frac{\alpha}{a_0^2}\right) \cdot a^2}{\frac{3}{4} \cdot \gamma \cdot \omega^2 \cdot a^2 + \alpha}. \tag{42}$$

According to Eq. (42), after a certain mathematical transformation, the result is the change of amplitude over time:

$$a^2 = \frac{\alpha}{\left(\frac{3}{4} \cdot \gamma \cdot \omega^2 + \frac{\alpha}{a_0^2}\right) \cdot e^{-b\alpha t} - \frac{3}{4} \cdot \gamma \cdot \omega^2}.$$

Respectively:

$$a = \sqrt{\frac{\alpha}{\left(\frac{3}{4} \cdot \gamma \cdot \omega^2 + \frac{\alpha}{a_0^2}\right) \cdot e^{-b\alpha t} - \frac{3}{4} \cdot \gamma \cdot \omega^2}}. \tag{43}$$

Substituting integral Eq. (36) into differential Eq. (35), the second dependency is obtained:

$$\begin{aligned} \frac{d\psi}{dt} &= \omega - \frac{b}{2\pi \cdot a \cdot \omega} \cdot \int_0^{2\pi} \big[-\alpha \cdot a \cdot \omega \cdot \sin\psi + \\ &\quad \beta \cdot a^2 \cdot \omega^2 \cdot \sin^2\psi - \gamma \cdot a^3 \cdot \omega^3 \cdot \sin^3\psi \big] \cdot \cos\psi \, d\psi; \\ \frac{d\psi}{dt} &= \omega - \frac{b}{2\pi \cdot a \cdot \omega} \cdot \big(-\alpha \cdot a \cdot \omega \cdot 0 + \\ &\quad \beta \cdot a^2 \cdot \omega^2 \cdot 0 + \gamma \cdot a^3 \cdot 0 \big) = \omega. \end{aligned} \tag{44}$$

According to Eq. (24), it follows that:

$$d\psi = \omega \cdot dt. \tag{45}$$

Values for initial conditions are $t = 0$, $\psi = 0$. According to Eq. (45), it follows that the definite integral is

$$\int_0^{\psi} d\psi = \int_0^{t} \omega \, dt. \tag{46}$$

From here, the final phase of oscillation will be

$$\psi = \omega \cdot t. \tag{47}$$

Using Eq. (30) and the solution from Eqs. (43) and (47), the law of motion is obtained:

$$z = \sqrt{\frac{4a_0^2 \cdot \alpha \cdot e^{b\alpha t}}{4\alpha + 3\gamma\omega^2 a_0^2 - 3\gamma\omega^2 a_0^2 e^{b\alpha t}}} \cdot \cos\omega t. \tag{48}$$

According to the initial coordinate system in Eq. (13), the final law of motion is

$$y = z + \frac{F_2}{m} \cdot \frac{\mu(V_o)}{\omega^2}; \tag{49}$$

$$y = \sqrt{\frac{4a_0^2 \cdot \alpha \cdot e^{b\alpha t}}{4\alpha + 3\gamma\omega^2 a_0^2 - 3\gamma\omega^2 a_0^2 e^{b\alpha t}}} \cdot \cos\omega t + \frac{F_2}{m} \cdot \frac{\mu(V_o)}{\omega^2}. \tag{50}$$

Constants α and γ are defined by Eq. (18). Notable is the fact that law of motion is independent from constant β.

In self-excited processes, as it is described, it is typical to find a steady state in which the amplitude a is constant under an unlimited increase of time. The simplest way of obtaining this amplitude is through expression of Eq. (37) when the result is equal to zero:

$$\frac{da}{dt} = 0. \tag{51}$$

Respectively:

$$\frac{ab}{2}\left(\alpha + \frac{3}{4}\gamma \cdot a^2 \cdot \omega^2\right) = 0. \tag{52}$$

From here it follows that

$$\alpha + \frac{3}{4}\gamma \cdot a^2 \cdot \omega^2 = 0.$$

Respectively:

$$a_{st} = \frac{2}{\omega}\sqrt{\frac{\alpha}{-3\gamma}}. \tag{53}$$

Solution in Eq. (53) can be also obtained using a different approach starting from expression for amplitude from Eq. (43):

$$a_{st} = \lim_{t\to\infty} a = \lim_{t\to\infty} \sqrt{\frac{4a_0^2 \cdot \alpha \cdot e^{b\alpha t}}{4\alpha + 3\gamma\omega^2 a_0^2 - 3\gamma\omega^2 a_0^2 e^{b\alpha t}}}. \tag{54}$$

Dividing the element under square root with $e^{b\alpha t}$:

$$a_{st} = \lim_{t\to\infty} \sqrt{\frac{4a_0^2 \cdot \alpha}{\frac{4\alpha + 3\gamma\omega^2 a_0^2}{e^{b\alpha t}} - 3\gamma\omega^2 a_0^2}}.$$

From here the amplitude of steady self-excited oscillations will be

$$a_{st} = \sqrt{\frac{4a_0^2\alpha}{-3\gamma\omega^2 a_0^2}} = \frac{2}{\omega}\sqrt{\frac{\alpha}{-3\gamma}}; \tag{55}$$

which is in agreement with solution from Eq. (53). It should be noted that amplitude a_{st} is not dependent on constant β. Substituting the coefficients α and γ according to Eq. (18) in Eq. (55), it follows:

$$a_{st} = \frac{2}{\omega}\sqrt{\frac{\frac{F_2 \cdot \mu'(V_o)}{b \cdot m} + \frac{1}{m}}{-3 \cdot \frac{F_2 \cdot \mu'''(V_o)}{6 \cdot b \cdot m}}}. \tag{56}$$

Respectively, the final form:

$$a_{st} = \frac{2}{\omega}\sqrt{\frac{-2 \cdot [F_2 \cdot \mu'(V_o) + b]}{F_2 \cdot \mu'''(V_o)}}. \qquad (57)$$

According to referenced data, experimental dependency of the friction coefficient from relative speed in a large number of cases is a polynomial of the third degree:[22,23]

$$\mu(V) = A \cdot V^3 + B \cdot V^2 + C \cdot V + D. \qquad (58)$$

The above squared element standing next to the B can be mostly disregarded because of its very small value ($B \approx 0$). Taking into consideration this fact, Eq. (58) can be interpreted with high accuracy as

$$\mu(V) = A \cdot V^3 + C \cdot V + D. \qquad (59)$$

From experimental dependencies in Fig. 3, it can be noted that in all cases, it applies that $A > 0$, $C < 0$, $D > 0$. This also follows from the shape of the experimental curve. From the condition of the minimum of function $\mu(V)$, it follows $A > 0$. With the application of repeated differentiation, Eq. (59) will be

$$\mu'(V) = 3 \cdot A \cdot V^2 + C; \qquad \mu''(V) = 6 \cdot A \cdot V;$$
$$\mu'''(V) = 6 \cdot A = const. \qquad (60)$$

The denominator in Eq. (57) is greater than zero. Taking into consideration that $\mu'''(V) > 0$, it follows that the numerator has to be positive. It will happen only in the case when

$$F_2 \cdot \mu'(V_o) + b < 0. \qquad (61)$$

Respectively:

$$\mu'(V_o) < -\frac{b}{F_2}; \qquad (62)$$

or because $b > 0$:

$$\mu'(V_o) < 0. \qquad (63)$$

From here it follows that Eq. (63) presents the condition for occurrence of self-excited oscillation for the observed model of the tool holder in the turning machining. The gradient of change for the friction coefficient curve $\mu'(V_o)$ is supposed to be negative, respectively, and the curve must have a descending character, as in Fig. 3; for instance, where $\mu'(V_o) > 0$ system is stable, and there is no possibility for self-excited oscillation occurrence. Unstable area is crosshatched on Fig. 3.

By simplification of the derived nonlinear differential Eq. (15), in a way that nonlinear elements are discarded, the linear equation is obtained:

$$\ddot{z} + \omega^2 z = \left[-\frac{F_2 \cdot \mu'(V_o)}{m} - \frac{b}{m}\right] \cdot \dot{z}. \qquad (64)$$

According to a theory of linear second-order differential equations with constant coefficients,[24] the condition of dynamic instability for observed model is obtained:

$$\frac{F_2 \cdot \mu'(V_o)}{m} + \frac{b}{m} < 0. \qquad (65)$$

Respectively:

$$F_2 \cdot \mu'(V_o) + b < 0;$$
$$\mu'(V_o) < -\frac{b}{F_2}; \qquad (66)$$

and finally:

$$\mu'(V_o) < 0. \qquad (67)$$

This is in agreement with previously obtained condition in Eq. (63).

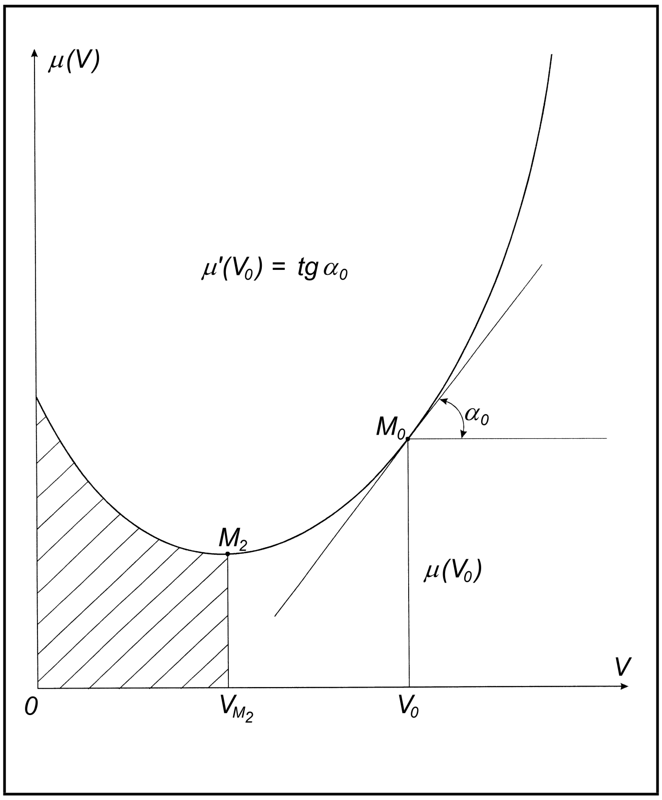

Figure 3. Characteristics of friction coefficient in relation with relative speed, with representation of unstable area.

4. CONCLUSIONS

For obtaining a solution for the required differential equation, there is no general procedure, so the problem is solved with the method of slowly varying coefficients. This method falls within a group of approximate methods because of the fact that solutions of a nonlinear problem do not exist in a final form.

General methods for solving nonlinear problems also do not exist. The main reason for this is the fact that general features that are valid in linear systems are not valid here. The observed problem of self-excited oscillation could be described through linear theory only roughly and the obtained solution would not have sufficient precision. The main reason for this is mostly that linearization drastically changes the system's structure and the structure of differential equations.

The obtained law of motion is in the form of complex Eq. (50), where larger numbers of influential factors are included. This law provides detailed quality analysis of the process and determining speed $\dot{y} = \frac{dy}{dt}$, respectively, and acceleration $\ddot{y} = \frac{d^2y}{dt^2}$.

As it can be seen based on the form of the experimental curve for the friction coefficient and relative speed dependency, self-excited oscillations are impossible in the area of ascending characteristics of the friction coefficient and vice versa. This conclusion can be also drawn from simplification of the model through linearization.

The amplitude of self-excited oscillations descend when the angular frequency is increasing, which follows from Eq. (55). Also, a greater damping factor causes a smaller amplitude of self-excited steady oscillation, which follows from the same equation. Characteristics of self-excited oscillations in the described model mostly depend on the character of the friction force. This conclusion is extracted from derived dependencies.

Angular frequency in the mentioned nonlinear oscillations depends on the amplitude, Eq. (57), and initial conditions of movement. This is not the case in free oscillations described with the linear model. The presented problem can be solved on the basis of the experimental function of friction using Eq. (58).

The derived general model based on the method of slowly varying coefficients can be directly applied in all cases where nonlinearity is not too large. This is usually the case in the field of machine tools. There is no major limitation in the model application. In general, the function F can have more complex character than the mentioned model; for example, $F = F(x, \dot{x})$. One of the most characteristic examples of the model application would be gear for the linear movement of machine tools where there is a friction on contact surfaces. According the theoretical analysis of the results, this paper is a base for further experimental research.

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